Denoising Poisson Noise

Joe Sadow
In Collaboration with: Dr. Toby Sanders

ARIZONA STATE UNIVERSITY
SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES

Nov. 13, 2017

0Picture Credit: Dr. Tron Omland
Poisson Distribution I

Given a binomial process where \( p \) – probability of success \( N \) – number of trials, the probability of \( k \) successes is:

\[
P(k) = \binom{N}{k} p^k (1 - p)^{N-k}
\]

As \( N \to \infty \),

\[
\binom{N}{k} = \frac{N!}{(N-k)!k!} \approx \frac{N(N-1)\ldots(N-k+1)}{k!} \approx \frac{N^k}{k!}
\]

Let

\[
\lambda = Np = \text{expected number of successes}
\]
Poisson Distribution II

For large $N$ and $\lambda = Np$:

$$P(k) \approx \frac{\lambda^k}{k!} (1 - p)^{\lambda/p}$$

As $p \to 0$ we have $(1 - p)^{1/p} \to e^{-1}$ to give the Poisson distribution:

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$  Mean and Variance : $\lambda$

Law of Rare Events: If the number of trials is big $N \to \infty$ and the probability of success is small $p \to 0$ then Poisson is appropriate
Poisson Noise is **signal dependent** with each signal point (mutually independent) $y_i \leftarrow \text{Poiss}(y_i)$

$$\text{SNR} = \frac{y_i}{\sqrt{y_i}} = \sqrt{y_i}$$
Applications of Poisson / Shot Noise

- Noise in images is usually **brightness** dependent
- Image sensors measure scene irradiance by counting the number of discrete photons
- As a result images in low light have a **low SNR**
Solving the Denoising Problem

Given a noisy image \( \tilde{f} \in \mathbb{R}^N \) we want to find a regularized version:

\[
\arg \min_f [\mu E(f, \tilde{f}) + R(f)]
\]

For some error metric \( E \) (fidelity) and smoothing enforcer (regularizer) \( R \) and tuning parameter \( \mu \)

Bayes Approach: Recast problem using Bayes Formula with an exponential transformation
Bayes Formulation (MLE)

We have:

\[ f^* = \arg \min_f [\mu E(f, \tilde{f}) + R(f)] \]

\[ = \arg \max_f [e^{-\mu E(f, \tilde{f})} e^{-R(f)}] \]

\[ = \arg \max_f P(f|\tilde{f}) \]

\[ = \arg \max_f P(\tilde{f}|f)P(f) \]

\[ \text{MSE} \]

Take \( E(f, \tilde{f}) = \frac{1}{2} \| f - \tilde{f} \|_2^2 \) then:

\[ P(\tilde{f}|f) = \prod_{i=1}^{N} e^{-\frac{\mu}{2} (f_i - \tilde{f}_i)^2} \sim \text{Normal} \]
Derivation of Poisson

Suppose the prior is Poisson with pixel-wise mean $b_i$, using Stirling's approximation:

$$P(\tilde{f}|f) = \prod_{i=1}^{N} \frac{(f_i)^{\tilde{f}_i}}{\tilde{f}_i!} e^{-f_i}$$

$$-\ln P(\tilde{f}|f) = \sum_{i=1}^{N} -\tilde{f}_i \ln(f_i) + \ln(\tilde{f}_i!) + f_i$$

$$= \sum_{i=1}^{N} -\tilde{f}_i \ln(f_i) + \tilde{f}_i \ln(\tilde{f}_i) - \tilde{f}_i + f_i$$

With Poisson data, the fidelity term is negative-Log Likelihood

$$E(f, \tilde{f}) = \sum_{i=1}^{N} f_i + \tilde{f}_i \ln(\tilde{f}_i) - \tilde{f}_i \ln(f_i) - \tilde{f}_i$$
Numerical Experiment Setup

With $R(f) = \|D^2 f\|_2^2$ (2nd order Tikhonov regularizer), solve:

$$F(f) = \mu \sum_{i=1}^{N} f_i + \tilde{f}_i \ln(\tilde{f}_i) - \tilde{f}_i \ln(f_i) - \tilde{f}_i + \|D^2 f\|_2^2$$

Using Gradient Descent:

$$f_{rec}^{(k+1)} = f_{rec}^{(k)} - \tau^{(k)} \nabla F(f_{rec}^{(k)})$$
Test Images

1) Background

2) Simulation and Results

Test Signal

Noisy Image

True

Original Image
1D Simulation Results

Poisson vs. LS with Tikhonov 2 Reg

$y = f(x)$

$x$-axis range: -1 to 1
$y$-axis range: 0 to 900
2D Simulation Results

Original Image

Noisy Image

LS

Poisson
2D Simulation Results
Choosing $\mu$: The Discrepancy principle

Given $b_i \sim \text{Poiss}(f_i)$, we enforce that the mean of this Poisson distribution equal its variance:

$$
\sum_{i=1}^{N} f_i = \sum_{i=1}^{N} E(b_i - f_i)^2
$$

Checking this condition allows periodic adjustments on the regularization strength which is necessary due to the signal-dependent nature of the noise.
Conclusions/ Ongoing and Future Efforts

Work Done so Far

- Confirmed derivation of Poisson objective function $E$
- Numerical results showing benefit of minimizing $E$

Ongoing and Future Work

- Choice of fidelity parameter $\mu$
- Choice of step length $\tau^{(k)}$ with backtracking
- Applying these tools to data seen in Mike’s talk
- PCA approach
Sources and Further Reading

1. *Image deblurring with Poisson data: from cells to galaxies* - Bertero, Boccacci, Desidera, and Vicidomini

2. *Inverse problems: A Bayesian perspective* - Stuart

3. *A Variational Approach to Reconstructing Images Corrupted by Poisson Noise* - Le, Chartrand, and Asaki

4. Theoretical results: *Total variation-penalized Poisson likelihood estimation for ill-posed problems* - Bardsley and Luttman
Thanks!