Motivation

In magnetic resonance (MR) imaging, image data is collected as Fourier coefficients, usually in non-Cartesian coordinates. For this reason, standard reconstruction methods cannot be implemented.

Example. Consider the following 1D signal $f \in L^2[-\pi, \pi]$. Say that we are given Fourier samples of $f$ as the linear functions

$$\{ \hat{f}(\lambda_n) \}_{n \in \mathbb{Z}}$$

where $\hat{f}(\lambda_n) = \hat{f}(\lambda)e^{-i\lambda_n x}$ for some family $\{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$. If we in fact have $\{\lambda_n\} = \mathbb{Z}$, then we can reconstruct $f$ exactly. However, in many instances this is not the case. Therefore, non-standard methods of reconstructing a signal from nonuniform samples are necessary.

Problem Statement

Let $H$ be a separable Hilbert space of functions on a domain $E$. The problem we have studied has two forms:

1. Construct (exactly, if possible) the signal $g \in H$ given only information in the form of the linear functionals $\{g, f_n\}_{n \in \mathbb{N}}$ for a fixed family $\{f_n\}_{n \in \mathbb{N}}$ in $H$.

2. Construct a family $\{f_n\}_{n \in \mathbb{N}}$ in $H$ so that every $g \in H$ can be reconstructed given the samples $\{g, f_n\}_{n \in \mathbb{N}}$.

In addition to accuracy and uniqueness, "good" solutions to both of these questions also give numerically efficient and practically implementable reconstruction algorithms.

Hilbert Frames

We are particularly interested in the case when sampling functions $\{f_n\}_{n \in \mathbb{N}}$ form a frame in $H$, that is, there exist positive constants $A$ and $B$ such that

$$A\|f\|^2 \leq \sum_{n \in \mathbb{N}} |\langle f, f_n \rangle|^2 \leq B\|f\|^2$$

for all $f \in H$.

Theorem. When $\{f_n\}_{n \in \mathbb{N}}$ is a frame for $H$, then the frame operator $S : H \rightarrow H$, defined by

$$Sf = \sum_{n \in \mathbb{N}} \langle f, f_n \rangle f_n$$

is positive, self-adjoint, and invertible.

As a result, a function can be reconstructed (or represented) exactly by its frame samples.

Induced Hilbert Spaces

As described by S. Saitoh, the given sampling sequence $\{f_n\}_{n \in \mathbb{N}}$ naturally generates a reproducing kernel Hilbert space in the following way:

Let $Z$ be any set, and $\Phi : Z \rightarrow H$ be a function. Denote by $F(Z)$ the set of scalar-valued functions on $Z$, and define the map $L : F(Z) \rightarrow \mathbb{C}$ by

$$L(f) = \langle f, \Phi \rangle$$

Note. Here, we take $Z = \mathbb{N}$ and $\Phi(n) = f_n$, so that $L : F \rightarrow (\{f_n\}_{n \in \mathbb{N}})$.

Theorem. (Saitoh, 1983)

The range of $L$, denoted $K$, is a Hilbert space under the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\|f\| = \sqrt{\langle f, f \rangle}$$

Furthermore, $K$ admits the reproducing kernel $K : Z \times Z \rightarrow H$ defined by

$$K(p, q) = \langle \Phi(p), \Phi(q) \rangle$$

Fact. If $\langle \Phi(p) \rangle$ is complete in $H$, then $L$ is an isometry (otherwise, $L$ will not be injective).

Lemma. Associated to every element $f \in K$ there is a unique $F^* \in K$ with $L(F^*) = f$ and which satisfies $\|F^*\| = \|f\|$. Provided that the sampled function $f \in K$ satisfies some regularization conditions, there is an explicit reconstruction algorithm given by

$$F^* = \sum_{n \in \mathbb{N}} \langle f, LE_n \rangle E_n$$

where $\{E_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for $H$.

Fact. Computation of the norm $\| F^* \|$ is a costly minimization process. For this reason, we hope to design the sampling scheme $\{\Phi(p)\}_{p \in Z}$ so that $\|f\|$ can be defined as an integral formula.

Fact. The family $\{\Phi(p)\}_{p \in Z} = \{L(K(p))\}_{p \in Z}$ is a frame in $H$ if and only if it is a frame sequence (a frame for its closed span).

Corollary. If $\{\Phi(p)\}_{p \in Z}$ is a frame sequence, then the function $\nu : K \rightarrow \mathbb{R}$ defined by

$$\nu(f) = \sum_{n \in \mathbb{N}} |\langle f, f_n \rangle|^2$$

is a norm on $K$ that is equivalent to the induced norm $\| f \|$.

Result. If $\{\Phi(p)\}_{p \in Z}$ is a Parseval frame sequence (a frame sequence with frame bounds $A = B = 1$), then the function $\nu(f)$ above in fact is an integral formula for the norm $\| f \|$.

Example. This is particularly applicable when the set $\{\Phi(p)\}_{p \in Z}$ is a family of translations of a single function, since these are the commonly-studied type of frame sequence in $L^2(\mathbb{R})$. Furthermore, the set $\{\phi_p\}_{p \in Z}$ corresponds to a frame of translations on $K$ which also form a reproducing kernel, and hence we expect $K$ to behave much like a shift-invariant space. Investigating this question is one of our current research projects.

Localization and Re-projection for Frames

Introduction. For $f \in H$, say that we are given a finite sequence of frame coefficients $\{(f_n)_{n \in \mathbb{N}}\}$, and a corresponding approximation $Sf$ for $f$. We now would like to re-project $Sf$ onto a 'better' family $\{g_n\}_{n \in \mathbb{N}}$, whereby we recover $f$ pointwise with high accuracy.

Technique. If $Sf$ is the finite-dimensional approximation of $f$ with respect to the family $\{f_n\}_{n \in \mathbb{N}}$, then the error with respect to the family $\{g_n\}_{n \in \mathbb{N}}$ is

$$\|f - T \Phi(Sf)\| \leq \|f - T \Phi(f)\| + \|T \Phi(f) - T \Phi(Sf)\|$$

where $T \Phi$ is the finite-dimensional approximation of $f$ with respect to $\{g_n\}$. We hope to generate good re-projections using the theory of localized frames.

Definition. (Gröchenig, 2001)

Given $s > 0$, the frame $\{f_n\}_{n \in \mathbb{N}}$ is $s$-localized with respect to the basis $\{g_n\}_{n \in \mathbb{N}}$ if there exists a positive constant $C$ such that

$$\|f_n\| \leq C(1 + [n - m]^{-s}) \quad \text{and} \quad \|f_n - g_n\| \leq C(1 + [n - m]^{-s})^{-1}$$

for all $n, m \in \mathbb{N}$, where $\{g_n\}$ is the dual basis to $\{g_n\}$. Similarly, the frame $\{f_n\}$ is self-localized if

$$\|f_n\| \leq C(1 + [n - m]^{-s})^{-1}$$

Question. If $Sf$ is the approximation of $f$ with respect to the frame $\{f_n\}_{n \in \mathbb{N}}$ and $\{f_n\}_{n \in \mathbb{N}}$ is $s$-localized with respect to the basis $\{g_n\}_{n \in \mathbb{N}}$, how accurate is the re-projection of $Sf$ onto this basis?

Similarly, given a frame $\{f_n\}_{n \in \mathbb{N}}$, it is possible to find a B-spline which localizes $\{f_n\}$, and how to what degree?

Example. When $\{f_n\}$ is the standard Fourier basis, it has been shown that a finite-dimensional approximation $Sf$ can be re-projected onto Gegenbauer polynomials to recover $f$ pointwise with exponential accuracy. We hope to generalize this to more general families using the theory of localized frames. Of particular interest are nonuniform frames of complex exponentials reprojected onto orthogonal polynomials (e.g. Gegenbauer, Hermite, Freud).

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