Theorem 1 Let $S \subset \mathbb{R}^n$ and define $H$ to be the set of all points $z \in \mathbb{R}^n$ such that for some positive integer $k$, some points $z_1, \ldots, z_k \in S$, and some positive numbers $t_1, \ldots, t_k$ with $\sum_{i=1}^{k} t_i = 1$

$$z = \sum_{i=1}^{k} t_i z_i.$$ 

Then $H$ is a convex hull of $S$.

Proof. It is enough to show the following three facts.

1. $H$ contains $S$

2. $H$ is convex

3. Every convex set containing $S$ must contain $H$

Part (1) is clearly true. Simply take $k = 1$ and $t_1 = 1$ to see that all points from $S$ are special cases of formula for $z$.

Part (2) requires more work. Let $x, y \in H$ and let $z = tx + (1 - t)y$. Since $x \in H$ and $y \in H$ there are points $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ all in $S$ and positive numbers $p_1, \ldots, p_r, q_1, \ldots, q_s$ such that

$$x = \sum_{i=1}^{r} p_i x_i, \sum_{i=1}^{r} p_i = 1 \quad (1)$$

and

$$y = \sum_{i=1}^{s} q_i y_i, \sum_{i=1}^{s} q_i = 1. \quad (2)$$

Let

$$z_i = x_i, i = 1 \ldots r,$$

$$z_{r+i} = y_i, i = 1 \ldots s$$

and

$$t_i = t p_i, i = 1 \ldots r,$$

$$t_{r+i} = (1 - t) q_i, i = 1 \ldots s.$$

Then

$$\sum_{i=1}^{r+s} t_i z_i = \sum_{i=1}^{r} t_i z_i + \sum_{i=r+1}^{s} t_i z_i =$$

$$= t \sum_{i=1}^{r} p_i x_i + (1 - t) \sum_{i=1}^{s} q_i y_i = tx + (1 - t)y = z.$$ 

Also

$$\sum_{i=1}^{r+s} t_i = t \sum_{i=1}^{r} p_i + (1 - t) \sum_{i=1}^{s} q_i = t + (1 - t) = 1.$$ 

Thus $z$ is a convex combination of points from $S$ and so $z \in H$. 
Now let us prove part (3). We shall do this by induction on $k$ in the definition of $z$.

- **(base case)** Case $k = 1$ is clear as points from $S$ by definition are in $C$.

- **(inductive step)** Assume all points $z$ with upper index in summation $k - 1$ are in $C$ and consider $z = \sum_{i=1}^{k} t_i z_i$ with $\sum_{i=1}^{k} t_i = 1$. Define $t = 1 - t_k$ and let

$$r_i = \frac{t_i}{t}, i = 1, \ldots, k - 1.$$ 

We have $0 < t < 1$ and

$$\sum_{i=1}^{k-1} r_i = \frac{1}{t} \sum_{i=1}^{k-1} t_i = \frac{1}{t}(1 - t_k) = 1.$$ 

Thus, by inductive assumption,

$$z' = \sum_{i=1}^{k-1} r_i z_i \in C.$$ 

Since $z_k \in S$, $z_k$ is also in $C$ and $C$ is a convex set thus it contains

$$tz' + (1 - t)z_k$$

but

$$tz' + (1 - t)z_k = t \sum_{i=1}^{k-1} r_i z_i + t_k z_k = \sum_{i=1}^{k} t_i z_i = z.$$ 

Therefore $z$ is in $C$. 