Relationships

**Fact 1**  *The dual of the dual is always the primal problem.*

**Fact 2**  *If the primal is unbounded then the dual is infeasible.*

**Fact 3**  *If the dual is unbounded then the primal is unbounded.*
Example 1

maximize $-x_1 - 2x_2$

subject to:

\[ -3x_1 + x_2 \leq -1 \]
\[ x_1 - x_2 \leq 1 \]
\[ -2x_1 + 7x_2 \leq -1 \]
\[ 9x_1 - 4x_2 \leq 6 \]
\[ -5x_1 + 2x_2 \leq -3 \]
\[ 7x_1 - 3x_2 \leq 6 \]
\[ x_1, x_2 \geq 0 \]
Example solution

Dual: \textbf{minimize} \(-y_1 + y_2 + 6y_3 + 6y_4 - 3y_5 + 6y_6\)
subject to:
\begin{align*}
-3y_1 + y_2 - 2y_3 + 9y_4 - 5y_5 + 7y_6 &\geq -1 \\
y_1 - y_2 + 7y_3 - 4y_4 + 2y_5 - 3y_6 &\geq -2 \\
y_1, y_2, y_3, y_4, y_5, y_6 &\geq 0
\end{align*}

We solve:
\textbf{maximize} \(y_1 - y_2 - 6y_3 - 6y_4 + 3y_5 - 6y_6\)
subject to:
\begin{align*}
3y_1 - y_2 + 2y_3 - 9y_4 + 5y_5 - 7y_6 &\leq 1 \\
-y_1 + y_2 - 7y_3 + 4y_4 - 2y_5 + 3y_6 &\leq 2 \\
y_1, y_2, y_3, y_4, y_5, y_6 &\geq 0
\end{align*}
Slack variables

\[ y_7 = 1 - 3y_1 + y_2 - 2y_3 + 9y_4 - 5y_5 + 7y_6 \]
\[ y_8 = 2 + y_1 - y_2 + 7y_3 - 4y_4 + 2y_5 - 3y_6 \]
\[ z = y_1 - y_2 - 6y_3 - 6y_4 + 3y_5 - 6y_6 \]

Pivot with \( y_5 \) entering and \( y_7 \) leaving:

\[ y_5 = \frac{1}{5} - \frac{3}{5}y_1 + \frac{1}{5}y_2 - \frac{2}{5}y_3 + \frac{9}{5}y_4 + \frac{7}{5}y_6 - \frac{1}{5}y_7 \]
\[ y_8 = \frac{12}{5} - \frac{1}{5}y_1 - \frac{3}{5}y_2 + \frac{31}{5}y_3 - \frac{2}{5}y_4 - \frac{1}{5}y_6 - \frac{2}{5}y_7 \]
\[ z = \frac{3}{5} - \frac{4}{5}y_1 - \frac{2}{5}y_2 - \frac{36}{5}y_3 - \frac{3}{5}y_4 - \frac{9}{5}y_6 - \frac{3}{5}y_7 \]

which is optimal!!!!

Read the optimal solution to the primal from the dictionary \( x_1 = \frac{3}{5}, x_2 = 0. \)
Example 2

Consider the following problem maximize

\[ 3x_1 + 2x_2 - x_3 \]

subject to:

\[ x_1 - x_2 + x_3 \leq -1 \]
\[ x_1 + x_2 \leq 1 \]
\[ x_2 + x_3 \leq 4 \]
\[ x_1, x_2, x_3 \geq 0 \]

and solution \((0, 1, 0)\) to the primal, \((1, 3, 0)\) to the dual. Are these solutions optimal?
Example solution

Dual:

\[
\begin{align*}
\text{minimize } & -y_1 + y_2 + 4y_3 \\
\text{subject to: } & \quad y_1 + y_2 \geq 3 \\
& \quad -y_1 + y_2 + y_3 \geq 2 \\
& \quad y_1 + y_3 \geq -1 \\
& \quad y_1, y_2, y_3 \geq 0
\end{align*}
\]

- (0, 1, 0) is a feasible solution to the primal.
- (1, 3, 0) is a feasible solution to the dual.
- Value of the objective functions of primal and dual are equal.

Solutions are optimal.