Linear Programming

Lecture 4

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Initialization

Given a problem (1):

**maximize** \( \sum_{j=1}^{n} c_j x_j \)

subject to

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, \ldots, m) \]
\[ x_j \geq 0 \]

consider problem (2):

**minimize** \( x_0 \)

subject to

\[ \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \quad (i = 1, \ldots, m) \]
\[ x_j \geq 0 \]

(1) has a feasible solution iff (2) has a feasible solution

with \( x_0 = 0 \)
What to do with (2)

- Change **minimize** $x_0$ to **maximize** $-x_0$.

- Get the first dictionary:
  
  $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} + x_0$
  
  $w = -x_0$

  This dictionary is infeasible (as long as some $b_i$’s are negative).

  Pivot with $x_0$ as entering variable and $x_{n+k}$ with the smallest $b_k$ as a leaving variable.
Example 1

**maximize** $x_1 - x_2 + x_3$

**subject to**

\[ \begin{align*}
2x_1 - x_2 + 2x_3 & \leq 4 \\
2x_1 - 3x_2 + x_3 & \leq -5 \\
-x_1 + x_2 - 2x_3 & \leq -1 \\
\end{align*} \]

$x_1, x_2, x_3 \geq 0$

**Auxiliary problem:**

**maximize** $-x_0$

**subject to**

\[ \begin{align*}
2x_1 - x_2 + 2x_3 - x_0 & \leq 4 \\
2x_1 - 3x_2 + x_3 - x_0 & \leq -5 \\
-x_1 + x_2 - 2x_3 - x_0 & \leq -1 \\
x_0, x_1, x_2, x_3 & \geq 0
\end{align*} \]
Example 2

\[ x_4 = 4 - 2x_1 + x_2 - 2x_3 + x_0 \]
\[ x_5 = -5 - 2x_1 + 3x_2 - x_3 + x_0 \]
\[ x_6 = -1 + x_1 - x_2 + 2x_3 + x_0 \]

\[ w = -x_0 \]
Example 3

\( x_0 \) -entering, \( x_5 \) -leaving

\[
\begin{align*}
x_0 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
x_4 &= 9 - 2x_2 - x_3 + x_5 \\
x_6 &= 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\
w &= -5 - 2x_1 + 3x_2 - x_3 - x_5
\end{align*}
\]
Example 4

$x_2$ -entering, $x_6$ -leaving

\begin{align*}
x_2 &= 1 + 0.75x_1 + 0.75x_3 + 0.25x_5 - 0.25x_6 \\
x_0 &= 2 - 0.25x_1 - 1.25x_3 + 0.25x_5 + 0.75x_6 \\
x_4 &= 7 - 1.5x_1 - 2.5x_3 + 0.5x_5 + 0.5x_6 \\
w &= -2 + 0.25x_1 + 1.25x_3 - 0.25x_5 - 0.75x_6
\end{align*}
Example 5

$x_3$ --entering, $x_6$ -leaving

$$x_3 = 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 - 0.8x_0$$
$$x_2 = 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 - 0.6x_0$$
$$x_4 = 3 - x_1 - x_6 + 2x_0$$

$$w = -x_0$$
Back to the original problem

Set $x_0 = 0$ and solve $z = x_1 - x_2 + x_3$

\[
x_3 = 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6
\]
\[
x_2 = 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6
\]
\[
x_4 = 3 - x_1 - x_6
\]

\[
z = -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6
\]
General Rule

General rule: if in process of solving the auxiliary problem it is possible to take $x_0$ as a leaving variable then we choose $x_0$.

Optimal dictionary for an auxiliary problem

1. $x_0$ is nonbasic and $w = 0$ - gives feasible dictionary for problem (1)
2. $x_0$ is basic and $w \neq 0$ - problem (1) is infeasible
3. $x_0$ is nonbasic and $w = 0$ - cannot happen if we observe the general rule.
Two-phase simplex algorithm

Phase 1: solve an auxiliary problem

Phase 2: if solution of phase 1 gives a feasible solution to (1) then solve the original problem
Fundamental Theorem

Theorem 1  Every LP problem in the standard form has the following properties.

- If it has no optimal solution then it is either infeasible or unbounded.
- If it has a feasible solution then it has a basic feasible solution.
- If it has an optimal solution then it has a basic optimal solution.