Doubly stochastic matrices

An $n$ by $n$ matrix $X = [x_{ij}]$ is called doubly stochastic if

- $\sum_{i=1}^{n} x_{ij} = 1$ for every $j = 1, \ldots, n$
- $\sum_{j=1}^{n} x_{ij} = 1$ for every $i = 1, \ldots, n$
- $x_{ij} \geq 0$ for every $i, j = 1, \ldots, n$

Doubly stochastic matrices with integer values are called permutation matrices.
Generalization of Konig’s Th

**Theorem 1**  *For every* $n \times n$ *doubly stochastic matrix* $X = [x_{ij}]$ *there is a permutation matrix* $X^* = [x^*_{ij}]$ *such that if* $x_{ij} = 0$ *then* $x^*_{ij} = 0$.

Konig’s theorem is a special case: boys are rows girls are columns if $i$ and $j$ know each other then $x_{ij} = 1/k$ otherwise it is zero.
Convex combinations

If \( t_1, \ldots, t_k \) are such that \( \sum t_i = 1 \) and \( t_i \geq 0 \) then
\[
M = t_1 M_1 + t_2 M_2 + \cdots + t_k M_k
\]
is called a convex combination of matrices \( M_1, \ldots, M_k \).

Fact 2  Convex combination of doubly stochastic matrices is a doubly stochastic matrix.

In particular:
(1)   Convex combination of permutation matrices is a doubly stochastic matrix.

Converse to (1) is also true!
Birkhoff- von Neumann Th

**Theorem 3**  Every doubly stochastic matrix is a convex combination of permutation matrices.