General LP problems

maximize \[ \sum_{j=1}^{n} c_j x_j \]

subject to:

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad (i \in I) \]

\[ \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad (i \in E) \]

\[ x_j \geq 0, \quad (j \in R) \]

$I$-inequalities, $E$-equalities, $R$-variables that are explicitly restricted.
If $j$ is not in $R$ then $x_j$ is called free. $F$ - set of $j$’s such that $x_j$ is free.
The dual problem

minimize $\sum_{i=1}^{m} b_i y_i$

subject to:

$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad (j \in R)$  
$\sum_{i=1}^{m} a_{ij} y_i = c_j, \quad (j \in F)$  
$y_i \geq 0, \quad (i \in I)$
Example

Find the dual of the following problem:

**maximize** $3x_1 + 2x_2 + 4x_3$

**subject to:**

$$x_1 + 4x_2 + 2x_3 = 3$$
$$2x_1 + 2x_2 + x_3 = 1$$
$$x_1 - x_2 + 2x_3 \geq 4$$
$$x_1 \leq 2, \quad 0 \leq x_2, \quad 0 \leq x_3 \leq 1$$
Example cont.

First write the problem in an equivalent form:

**maximize** $3x_1 + 2x_2 + 4x_3$

**subject to:**

\[
\begin{align*}
x_1 + 4x_2 + 2x_3 &= 3 \\
2x_1 + 2x_2 + x_3 &= 1 \\
-x_1 + x_2 - 2x_3 &\leq -4 \\
x_1 &\leq 2 \\
x_3 &\leq 1 \\
x_2 &\geq 0, \ x_3 \geq 0
\end{align*}
\]
Example cont.

\[ R = \{2, 3\}, \quad F = \{1\} \] and so the dual is

\[ \text{minimize} \quad 3y_1 + y_2 - 4y_3 + 2y_4 + y_5 \]

subject to:

\[
\begin{align*}
y_1 + 2y_2 - y_3 + y_4 &= 3 \\
4y_1 + 2y_2 + y_3 &\geq 2 \\
2y_1 + y_2 - 2y_3 + y_5 &\geq 4 \\
y_3, y_4, y_5 &\geq 0
\end{align*}
\]
Weak duality theorem

**Fact 1**  For any solution $\bar{x}_1, \ldots, \bar{x}_n$ to the primal and any solution $\bar{y}_1, \ldots, \bar{y}_m$ to the dual we have

$$\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{i=1}^{m} b_i \bar{y}_i.$$
Duality theorem

**Theorem 2** If a linear programming problem has an optimal solution, then its dual has an optimal solution and the optimal values of two problems coincide.