Chapter 5
Relations

Definition 1 Let $A$ and $B$ be sets. A binary relation $R$ from $A$ to $B$ is any subset of $A \times B$.

If $A = B$ then a relation from $A$ to $B$ is called a relation on $A$.

Examples

- A relation $R$ on $N$ defined by $aRb$ when $a + b > 1$.

- Relations $\leq, \geq, =, <, >$ on $N$. 
• Relation $|$ on $\mathbb{N}$.

• Relation $\approx$ on subsets of $\mathbb{R}$.

• Relation $\perp$ on the set of all straight lines in the plane.

• Empty relation, relation $\mathbb{N} \times \mathbb{N}$ on $\mathbb{N}$.

• Relation on the set of all people in the world given by $aRb$ when $a$ knows $b$. 
Properties of relations
Let $R$ be a relation on $A$.

- $R$ is called **reflexive** if for every $a \in A$
  $$aRa.$$  

- $R$ is called **irreflexive** if for every $a \in A$,  
  $$\neg aRa.$$  

- $R$ is called **symmetric** if for every $a, b \in A$
  $$aRb \rightarrow bRa.$$
• $R$ is called **asymmetric** if for every $a, b \in A$,

\[ aRb \rightarrow \neg bRa. \]

• $R$ is called **antisymmetric** if for every $a, b \in A$,

\[ (aRb \land bRa) \rightarrow (a = b). \]

• $R$ is called **transitive** if for every $a, b, c \in A$,

\[ (aRb \land bRc) \rightarrow aRc. \]
Examples

- The empty relation on \( N \) and \( N \times N \) on \( N \).

- The relation of divisibility, \( | \), on \( N \).

- The relation of less than or equal to, \( \leq \), on \( R \).

- The relation of equality, \( = \), on \( R \).

- The relation of equipollence, \( \approx \), on \( P(R) \).
The relation of set inclusion, $\subseteq$, on $P(N)$.

**Examples**

- $R$ on $\mathbb{Z}$ defined by $aRb$ if $a - b$ is even.

- $R$ on $\mathbb{Z}$ defined by $aRb$ if $a + b$ is odd.

- $R$ on $\mathbb{Z}$ defined by $aRb$ if $a^2 - b^2 \geq 0$.

- $R$ on $P(\mathbb{Z})$ defined by $aRb$ if $a \cap b'$ is non-empty.
Directed graphs

- Reflexivity
- Symmetry
- Transitivity
- Irreflexivity
- Antisymmetry
- Asymmetry
**Equivalence relation**

**Definition 2** Let $R$ be a relation on $A$. Then $R$ is called an equivalence relation if it is reflexive, symmetric, and transitive.

**Example 1** Consider $R$ on $\mathbb{Z}$ given by $(a, b) \in R$ if $5 | (a - b)$.

**Example 2** Let $F$ be the set of all function $f : \mathbb{N} \to \{0, 1\}$. For $f, g \in F$, we say $f \sim g$ if $f(n) \neq g(n)$ for finitely many $n \in \mathbb{N}$.

**Example 3** Let $F$ be a set. The relation of equipollence on $P(F)$.
**Definition 3** Let $R$ be an equivalence relation on $A$ and let $a \in A$. The equivalence class of $a$, $[a]$ is the set

$$[a] = \{x | aRx\}.$$  

**Example 4** Consider $R$ on $\mathbb{Z}$ given by $(a, b) \in R$ if $5|(a - b)$. Find $[-31]$.

**Example 5** Let $\mathcal{F}$ be the set of all function $f : N \rightarrow \{0, 1\}$. For $f, g \in \mathcal{F}$, we say $f \sim g$ if $f(n) \neq g(n)$ for finitely many $n \in N$. Let $f$ be function which is always equal to 0. Find $[f]$.

**Example 6** Consider the equipollence relation on $P(R)$. Is $(0, 5) \in [(2, 9)]$?
Theorem 1 Let $R$ be an equivalence relation on a set $A$.

- For every $a \in A$, $a \in [a]$.

- For any $a, b \in A$, $aRb$ if and only if $[a] = [b]$.

- If $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$.

Definition 4 A partition of a non-empty set $A$ is a family of non-empty sets $\mathcal{F}$ such that (a) $X \cap Y = \emptyset$ for any two distinct sets $X, Y \in \mathcal{F}$ and (b) $\bigcup \mathcal{F} = A$. 
Theorem 2 Let $A$ be a non-empty set.

- If $R$ is an equivalence relation on $A$ then the family of equivalence classes is a partition of $A$.

- For a partition $\mathcal{F}$ of $A$ the relation $R$ given by $aRb$ when there is a set $X \in \mathcal{F}$ such that $x, y \in X$ is an equivalence relation on $A$.

Theorem 3 Let $A$ be a non-empty set. Let $\mathcal{E}$ be the set of all equivalence relations on $A$ and let $\mathcal{P}$ be the set of all partitions of $A$. Then $\mathcal{E} \approx \mathcal{P}$. 
Order relations

**Definition 5** A relation $R$ on $A$ is called a *partial order* if it is reflexive, transitive, and antisymmetric. A relation $R$ on $A$ is called a *linear order (total order)* if it is a partial order and for every $a, b \in A$ either $aRb$ or $bRa$ *(trichotomy law).*

**Example 7** We have:

- $\leq$ on $R$ is a total order;
- $\subseteq$ on $P(N)$ is a partial order but is not a total order.
Definition 6  Let $R$ be a partial order relation on set $A$ and let $A$ be a subset of $A$. An $x \in A$ is an upper bound of $A$ if for every $a \in A, aRx$. An $x \in A$ is called the least upper bound of $A$ if it is an upper bound and for every upper bound $y$ of $A, xRy$.

Definition 7 A total order $R$ on $A$ is said to satisfy the completeness property if every nonempty subset of $A$ which has an upper bound has the least upper bound.

Example 8  • Consider $\subseteq$ on $P(N)$. Find at least two upper bounds of $A = \{\{2\}, \{4\}, \{6\}\}$.

• Find at least two upper bounds of $A = (0, 1]$. 
• Consider $\leq$ on $\mathbb{R}$. Find the least upper bound of $A$ if one exists for $A = (0, 1]$, $A = (0, 1] \cup \{2, 7\}$, $A = (0, 1)$.

**Theorem 4** The relation $\leq$ on $\mathbb{Z}$ is a total order which satisfies the completeness property.

**Theorem 5** The relation $\leq$ on $\mathbb{Q}$ is a total order which does not satisfy the completeness property.

**Theorem 6** The relation $\leq$ on $\mathbb{R}$ is a total order which satisfies the completeness property.
**Axiom of Choice**

**Axiom of Choice:** For any collection of non-empty sets $\mathcal{A}$ there is a function $f : \mathcal{A} \rightarrow \bigcup \mathcal{A}$ such that for every $S \in \mathcal{A}$, $f(S) \in S$.

**Example 9**

- If $\mathcal{A}$ is finite then we can order the element in $\bigcup \mathcal{A}$ and define $f(S)$ to be the smallest element of $S$.

- Similarly if $\mathcal{A} = P(N) - \{\emptyset\}$ then $f(S)$ can be the smallest element of $S$.

- If $\mathcal{A}$ is the set of all non-empty open intervals then $f((a, b)) = (a + b)/2$. 
Example 10 (Prisoners Puzzle)  Denumerably many prisoners lined up want to guess the colors of their hats. No communication during the experiment is allowed but they can meet a day before to design a strategy. If only finitely many are wrong they will be set free.
Example 11 (Banach-Tarski Paradox) Paradoxical decomposition of a sphere:

- Cut into finitely many strange pieces
- Rotate and translate pieces to assemble two spheres which are like the original one
”To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed.”  Bertrand Russell, 1872-1970
Sequences

Definition 8 A sequence is a function \( a : N \to R \).

- We write \( a_n \) for the value \( a(n) \).
- We use \( \{a_n\}_{n=1}^{\infty} \) to denote a sequence.
Definition 9 We say that the sequence \( \{a_n\}_{n=1}^\infty \) tends to the limit \( a \) (written \( \lim_{n \to \infty} a_n = a \)) if

\[
\forall \epsilon > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0 \ |a_n - a| < \epsilon.
\]

Example 12

- \( \lim_{n \to \infty} \left(5 - \frac{2}{n}\right) = 5 \)

- \( \lim_{n \to \infty} \frac{2n^2 + 1}{n^2} = 2 \)

- \( \lim_{n \to \infty} (-1)^n \) does not exist.
Example 13 Show that if \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} b_n = b \) then \( \lim_{n \to \infty} a_n + b_n = a + b \).

Example 14 A sequence \( \{a_n\}_{n=1}^\infty \) is called monotone if \( a_n \leq a_m \) if \( n < m \). A sequence \( \{a_n\}_{n=1}^\infty \) is called bounded if there is a number \( M \) such that for every \( n \), \( |a_n| \leq M \). Show that if \( \{a_n\}_{n=1}^\infty \) is monotone and bounded then it has a limit.
Definition 10 We say that the sequence \( \{a_n\}_{n=1}^{\infty} \) is a Cauchy sequence if

\[
\forall \epsilon > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0 \forall m \geq n_0 |a_n - a_m| < \epsilon.
\]

Theorem 7 Let \( \{a_n\} \) be a sequence.

- If \( \{a_n\}_{n=1}^{\infty} \) is a Cauchy sequence (in \( \mathbb{R} \)) then \( \lim_{n \to \infty} a_n \) exists.

- If \( \lim_{n \to \infty} a_n \) exists then \( \{a_n\}_{n=1}^{\infty} \) is Cauchy.
Definition 11 We say that $a$ is an accumulation point of the sequence $\{a_n\}_{n=1}^{\infty}$ if

$$\forall \epsilon > 0 \forall n \in \mathbb{N} \exists m \geq n |a_m - a_0| < \epsilon.$$ 

Example 15 Show that if $\lim_{n \to \infty} a_n = a$ then $a$ is an accumulation point of $\{a_n\}_{n=1}^{\infty}$. Show that, the converse does not need to be satisfied.
The end