Basic properties and terminology.

1. Vector spaces
   - \( \mathbb{R}^n \) is the space of \((x_1, \ldots, x_n)\) with scalar multiplication and addition defined as in matrix algebra.
   - Abstract vector space: set \( V \) with addition and scalar multiplication which is closed under the operations and such that operations satisfy natural properties (a)-(h) (page 165).

2. Subspaces
   - \( W \) is a subspace of \( V \) if and only if (a) \( u + v \) is in \( W \) whenever \( u, v \) are in \( W \) (b) \( cv \) is in \( W \) whenever \( v \) is in \( W \) and \( c \) is a scalar.
   - Set of vectors which are solutions to a homogeneous system of equation forms a subspace (with addition and scalar multiplication defined as for matrices).

3. Linear combinations and independence
   - Linear combination of \( v_1, \ldots, v_n \) is any vector of the form \( c_1v_1 + c_2v_2 + \cdots + c_nv_n \).
   - Vectors \( v_1, \ldots, v_n \) are called linearly independent if \( c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0 \) implies \( c_1 = c_2 = \cdots = c_n = 0 \).
   - \( \text{span}(S) \) is the set of all linear combinations of vectors from \( S \).

4. Facts
   - For any set \( S \) of vectors from a vector space \( V \), \( \text{span}(S) \) is a subspace of \( V \).
   - Vectors \( v_1, \ldots, v_k \) in \( \mathbb{R}^n \) are linearly independent if and only if \( A = [v_1 \, v_2 \, \ldots \, v_k] \) has a \( k \times j \) submatrix with a non-zero determinant.

5. Bases
   - Set \( S \) of vectors is a basis of vector space \( V \) if (a) vectors from \( S \) are linearly independent and (b) \( \text{span}(S) = V \)
   - Standard basis of \( \mathbb{R}^n \): \( e_1 = (1, 0 \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1) \).
• If a basis has \( n \) vectors then any set of \( m > n \) vectors is linearly dependent.

• Any two bases have the same size. The size of a basis is called the dimension of the vector space.

• Let \( V \) be a vector space of dimension \( n \).
  – If \( S \) consists of \( n \) vectors which are linearly independent then \( S \) is a basis.
  – If \( \text{span}(S) = V \) and \( S \) consists of \( n \) vectors then \( S \) is a basis of \( V \).