More advanced topics.

1. Row and Column spaces

• Let \( A \) be \( m \) by \( n \). Basis of \( \text{Row}(A) \) contains non-zero rows of the echelon form of \( A \).

• Let \( A \) be \( m \) by \( n \). Basis of \( \text{Col}(A) \) contains columns of \( A \) which correspond to pivot columns of the echelon form of \( A \).

• We have

\[
\text{dim}(\text{Col}(A)) = \text{dim}(\text{Row}(A))
\]

which is called the rank of \( A \) (\( \text{rank}(A) \)) and

\[
\text{rank}(A) + \text{dim}(\text{Null}(A)) = n.
\]

2. Orthogonality

• Let \( V \) be a subspace of \( \mathbb{R}^n \). Then \( V^\perp \) contains all vectors \( u \) which are orthogonal to every vector from \( V \).

• Small facts about orthogonality:
  
  – If \( v_1, \ldots, v_k \) are mutually orthogonal then they are linearly independent.
  
  – \( (V^\perp)^\perp = V \).
  
  – The only vector in both \( V \) and \( V^\perp \) is 0.
  
  – If \( S \) is such that \( V = \text{span}(S) \) then \( u \) is in \( V^\perp \) if and only if \( u \) is orthogonal to every vertex from \( S \).

• We have

\[
\text{Row}(A) = \text{Null}(A)^\perp.
\]

and as a corollary

\[
\text{dim}(V) + \text{dim}(V^\perp) = n
\]

for every subspace \( V \) of \( \mathbb{R}^n \).

3. Normal systems and least squares
• If $Ax = b$ is inconsistent we can try to solve an approximate system that is we can project $b$ on $Col(A)$ to find vector $p$ in the solution space which is closest to $b$.

• Normal system: $A^T A \bar{x} = A^T b$. Least squares solution is $\bar{x}$ and $p := A \bar{x}$.

• If $A$ is $m \times n$ of rank $n$ then $A^T A$ is non-singular.

4. Gram-Schmidt algorithm

• Start with $v_1, \ldots, v_k$ which are linearly independent.

• Set $u_1 := v_1$ and then iterate and compute

$$u_{i+1} := v_{i+1} - \left( \frac{v_{i+1} \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{v_{i+1} \cdot u_i}{u_i \cdot u_i} u_i \right).$$

• If $v_1, \ldots, v_k$ are mutually orthogonal then projection of $b$ on $V := \text{span}(v_1, \ldots, v_k)$ is

$$p := \frac{b \cdot v_1}{v_1 \cdot v_1} + \cdots + \frac{p \cdot v_k}{v_k \cdot v_k}.$$

5. Eigenvalues and eigenvectors

• Eigenvalue $\lambda$, eigenvector $v$: $Av = \lambda v$ for nonzero $v$.

• Eigenvalues of $A$ are the roots of $\det(A - \lambda I) = 0$ (characteristic equation).

• To find eigenvector associated with $\lambda$ solve $(A - \lambda I)x = 0$. Eigenvectors associated with $\lambda$ form a subspace called the eigenspace.

6. Diagonalization

• Matrices $A$ and $B$ are similar if $B = P^{-1} AP$.

• An $n \times n$ matrix $A$ is similar to diagonal matrix $D$ if and only if $A$ has $n$ linearly independent eigenvectors.

(*) Eigenvectors associated with different eigenvalues are linearly independent.

(*) If an $n \times n$ matrix $A$ has $n$ distinct eigenvalues that it is similar to a diagonal matrix.

(*) Not covered in class.