Theorem 1 (Vizing, 7.1.10) If $G$ is a simple graph then $\chi'(G) \leq \Delta(G) + 1$.

Proof. Starting Point: Let $f$ be a proper $\Delta(G) + 1$-edge coloring of a subgraph $G'$ of $G$. We will show that if there is an un-colored edge of $G$ (i.e. $G' \neq G$) then we can augment the coloring and color the edge (by possibly recoloring some previous edges). Since the number of colors is $\Delta(G) + 1$ each vertex has a color which is not used on edges incident to it.

Downshifting idea: Suppose $uv$ is un-colored, color $a_0$ is missing at $u$, and color $a_1$ is missing on $v$. We may assume $a_1$ appears in $u$ on some edge $uv_1$ as otherwise we can color $uv$ with $a_1$. Let $a_2$ be the color missing on $v_1$. We can assume $a_2$ appears on some edge $uv_2$ as otherwise we can re-assign the colors:

$$f(uv_1) := a_2, f(uv) = a_1.$$

We can iterate this process and so we select $v_i$ so that $a_i$ is used on $uv_i$ and pick an un-used color $a_{i+1}$ on $v_i$ If $a_{i+1}$ is missing on $u$ then we can re-color by

$$f(uv_i) := a_{i+1}, f(uv_{i-1}) := a_i, f(uv_{i-2}) := a_{i-1}, \ldots, f(uv) := a_1.$$

This is called downshifting from $v_{i+1}$. Since there are only $\Delta(G) + 1$ colors some color will eventually will be repeated or we will downshift.

Looking at the repeated $a_k$ and $a_0, a_k$-alternating path: Let $l$ be the smallest index such that a color missing at $v_l$ appears on $a_1, \ldots, a_l$ and suppose the color is $a_k$.

- Color $a_k$ is missing on $v_l$ and on $v_{k-1}$. Color $a_k$ does appear on $uv_k$.
- If $a_0$ does not appear at $v_l$ then we downshift from $v_l$ using color $a_0$ and so assume $a_0$ appears at $v_l$.

Let $P$ be a maximal alternating path of edges colored $a_0$ and $a_k$ that begins at $v_l$ (with edge colored by $a_0$). As the coloring is proper there is obviously one such path. There are a few possibilities to consider.

1. If $P$ ends in $u$ then it reaches $u$ by an edge colored with $a_k$ as $a_0$ is missing at $u$ and so this is the edge $v_ku$. Then we downshift from $v_k$ and interchange colors on $P$.

2. If $P$ ends in $v_{k-1}$ then it ends in color $a_0$ as by definition $a_k$ is missing at $v_{k-1}$. Then downshift from $v_{k-1}$ and color $uv_{k-1}$ with color $a_0$ and interchange colors on $P$.  

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3. If $P$ ends in a vertex $w$ such that $w \neq u$ and $w \neq u_{k-1}$ then $w \notin \{u, v_{k-1}, v_k, v_l\}$. Then we downshift from $a_l$, color $uv_l$ with $a_0$ and interchange colors on $P$. 

Figure 1: Vizing’s Theorem