1 Tutte’s Theorem

Theorem 1 (Tutte, 3.3.3) A simple graph $G$ has a 1-factor if and only if $o(G - S) \leq |S|$ for every $S \subseteq V(G)$.

Necessity If $G$ has a 1-factor $M$ and $S \subseteq V(G)$ then every odd component of $G - S$ has at least one vertex which is saturated by an edge of $M$ with the second endpoint in $S$. As $M$ is a matching, $o(G - S) \leq |S|$.

Sufficiency We will prove it by contradiction.

• (We may assume that $G$ is a maximal counterexample.) Let $G' = G + e$, i.e. we add edge $e$ to $G$. Note that if $G'$ does not have a 1-factor then $G$ does not have 1-factor. Also $o(G' - S) \leq o(G - S)$ as adding $e$ cannot create new odd component in $G - S$. Consequently we may assume that $G$ is a maximal counterexample in the following sense. It has no 1-factor but adding any edge to $G$ gives a 1-factor and the Tutte’s condition for $G$ is satisfied.

• $(U)$ Let $U$ be the set of vertices in $G$ of degree $|V(G)| - 1$.

• (Case 1:) $G - U$ consists of disjoint complete graphs. Then $o(G - U) \leq |U|$ and in each even component we can find a 1-factor in each odd component we can find a matching that saturates all but one vertex. The remaining vertex can be paired with a vertex from $U$. What remains of $U$ must be even and so we can pair the remaining vertices as well.

• (Case 2:) $G - U$ is not a disjoint union of cliques.
Proof of Tutte’s Theorem (Case 2), $u, v, w, z$

- **($u, v, w$)**. First observe that there exist two vertices $u, v$ in $G - U$ which are at distance equal to 2 in $G - U$. Indeed let $u, v$ be two nonadjacent vertices in the same component which is not a complete graph. Consider the shortest path $u, w, x, \ldots, v$. Then $u, x$ are at distance two and are both connected with $w$. Assume $uw, vw \in E$ and $u$ are $v$ are not connected.

- **($z$)**. Second observe that there is a vertex $z \in G - U$ which is not connected to $w$. Indeed, otherwise $w$ is joint be an edge to every vertex in $G - U$ and to every vertex from $U$ (by definition of $U$) and so $w$ would be in $U$ which is not the case.

- **($M_1 \Delta M_2$)**. Since adding an edge to $G$ gives a 1-factor, $G + uv$ has a 1-factor $M_1$ and $G + wz$ has a 1-factor $M_2$. We will show that $M_1 \Delta M_2$ has a 1-factor that avoids $uv$ and $wz$. This gives a 1-factor in $G$.

- **($F$ and properties of $F$)**. Let $F = M_1 \Delta M_2$. Note that $uv, wz \in F$ as each is in only one of the $M_i$’s. Moreover every vertex of $G$ has degree in $F$ which is either 0 or 2 as it has degree exactly one in each of the $M_i$’s. Thus components of $F$ are even cycles and isolated vertices.

- **($C$ and easy case)**. Let $C$ be the component that contains $uv$. If $C$ does not contain $wz$ then we easily obtain a 1-factor of $C$ that avoids $uv$ and then do the same for the cycle that contains $wz$.

- **(Harder case)**. Suppose $C$ contains both $uv$ and $wz$. Recall that $uw, vw \in E(G)$ and we will use that information to construct a
new matching. Consider the path $S_1$ in $C$ between $w$ and $u$ and $S_2$ between $w$ and $v$ so that $C := S_1, uw, S_2$ and suppose without loss of generality that $z$ is in $S_2$. Consider the matching $N_2 := E(S_2) \cap M_1$, $N_1 := E(S_1) \cap M_2$ and let $N := N_1 \cup N_2 \cup \{wu\}$. $N \cup (M_1 - E(C))$ is a 1-factor in $G$. 