MAT 512 Review

1. Counting functions and ordered functions

2. Distribution problems

3. Recurrence relations

4. The pigeonhole principle: Erdős-Szekeres Theorem

5. Stirling numbers of the 1st kind
   - \( s(n + 1, k) = -ns(n, k) + s(n, k - 1) \)
   - \( (x)_n = \sum_{k=1}^{n} s(n, k)x^k \)

6. Stirling numbers of the 2nd kind
   - \( S(n + 1, k) = kS(n, k) + S(n, k - 1) \)
   - \( x^n = \sum_{k=1}^{n} S(n, k)(x)_k \)
   - Generating function \( S_k(x) = \frac{x^k}{(1-kx)(1-(k-1)x)\cdots(1-x)} \)
   - \( S(n, k) = \sum_{j=1}^{k} (-1)^{k-j} \frac{j^n}{j!(k-j)!} \)

7. Bell numbers
   - \( B_n = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i-1} B_{n-i} \)
   - The exponential generating function \( B(x) = e^{e^x-1} \)
   - \( B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} \)

8. Derangements
   - \( d(n) = (n - 1)(d(n - 2) + d(n - 1)) \)
   - The exponential generating function \( D(x) = \frac{e^{-x}}{1-x} \)
   - \( d(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \)

9. Catalan numbers
   - \( C_0 = 0, C_1 = 1 \) and \( C_n = \sum_{i=1}^{n-1} C_i C_{n-i} \)
   - The generating function \( C(x) = \frac{1}{2}(1 - \sqrt{1 - 4x}) \)
   - \( C_n = \frac{1}{n} \binom{2n-2}{n-1} \)
10. Generating functions

- Solving recurrence relations
- Generating function for the number of solutions \( q_m \) to \( x_1 + \cdots + x_n = m \)
- Using generating functions to prove identities.

11. The principle of inclusion-exclusion

\[
N_0 = \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|} |A_I|
\]

\[
N_p = \sum_{k=p}^{n} (-1)^{k-p} \binom{k}{p} \sum_{|J| = k} |A_J|
\]

- Counting arrangements of couples in a row (in a circle) etc.
- Counting solutions to equation \( x_1 + \cdots + x_n = m \) with \( l_i \leq x_i \leq u_i \).
- Counting the number of functions with exactly \( k \) fixed points (solutions with exactly \( k \) \( x_i \geq u \)) etc.
- Proving combinatorial identities.