Chapter 4
Linear Transformations
**Definition 1** A function $L$ from a vector space $V$ to a vector space $W$ is said to be a linear transformation if

$$L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)$$

for any $v, w \in V$ and any scalars $\alpha, \beta$.

**Fact 1** A function $L : V \rightarrow W$ is a linear transformation if and only if

- $L(v + w) = L(v) + L(w)$ for any $v, w \in V$.
- $L(\alpha v) = \alpha L(v)$ for any scalar $\alpha$ and any $v \in V$. 

Remark: A linear transformation from $V$ to $V$ is called a linear operator.

Examples:

- $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L((x_1, x_2)^T) = (x_1, -x_2)^T$.

- $L : \mathbb{R}^m \to \mathbb{R}^n$, $L(x) = Ax$ where $A$ is an $n \times m$ matrix.

- $L : \mathbb{R}^2 \to \mathbb{R}^3$, $L((x_1, x_2)^T) = (x_2, x_1, x_1 + x_2)^T$ as $L(x) = Ax$ with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$
Fact 2 Let $L : V \rightarrow W$ be a linear transformation. Then

- $L(0_v) = 0_w$.

- $L(-v) = -L(v)$.

- $L(\sum_{i=1}^{n} \alpha_i v_i) = \sum_{i=1}^{n} \alpha_i L(v_i)$.

Definition 2 Let $L : V \rightarrow W$ be a linear transformation.

- The kernel of $L$, $\text{ker}(L) = \{v \in V | L(v) = 0_W\}$. 
• For a subspace $S$ of $V$, the image of $S$, $L(S) = \{w \in W | w = L(v) \text{ for some } v \in S\}$. $L(V)$ is called the range of $L$.

**Theorem 3** Let $L : V \rightarrow W$ be a linear transformation and let $S$ be a subspace of $V$. Then $\ker(L)$ is a subspace of $V$ and $L(S)$ is a subspace of $W$.

**Example 1** Let $D : P_n \rightarrow P_n$ be given by $D(f) = f'$ and let $L : P_n \rightarrow P_{n+1}$ be given by $L(f(x)) = \int_0^x f(t)dt$. Find $\ker(D), \ker(L)$. Find $D(P_2), L(P_2)$.

**Matrix Representation of a Linear Transformation**
Theorem 4 If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then there is an $m \times n$ matrix $A$ such that

$$L(x) = Ax.$$ 

Moreover, for $j = 1, \ldots, n$, the $j$th column of $A$, $a_j = L(e_j)$.

Notation: Let $E = [v_1, \ldots, v_n]$ be an ordered basis in a vector space $V$. Then $v \in V$ can be written uniquely as

$$v = c_1v_1 + \cdots + c_nv_n.$$ 

Then

$$[v]_E = [c_1, \ldots, c_n]^T.$$
Theorem 5 Let $V$ be a vector space with an ordered basis $E = [v_1, \ldots, v_n]$ and let $W$ be a vector space with an ordered basis $F = [w_1, \ldots, w_m]$. Then for a linear transformation $L : V \to W$ there is an $m \times n$ matrix $A$ such that

$$[L(v)]_F = A[v]_E.$$ Moreover $a_j = L(v_j)$.

Example 2

- Find the matrix representation of $D : P_n \to P_{n-1}$ with standard bases in $P_n$.

- Find the matrix representation of $D : P_n \to P_n$ with standard bases in $P_n$. 
• Find the matrix representation of the integral $I : P_n \to P_{n+1}$ with standard bases in $P_n, P_{n+1}$.

**Example 3** Write a simple animation program in which the triangle $T$ with vertices $(0, 0), (1, 1), (1, -1)$ is rotating around $(0, 0)$.

• Represent $T$ using the $2 \times 4$ matrix $A_T$ where for two consecutive columns there is a line segment between them. This gives

\[
A_T = \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0
\end{pmatrix}.
\]

• We will be rotating points around $(0, 0)$ incrementing angel $\phi$ (using a small increment and reducing modulo 360).
Rotation can be accomplished by multiplying \( A_T \) by a rotation matrix \( R_\phi \) (from the left).

Example 4 (Cauchy’s functional equation) \( \text{Clearly if } f : R \rightarrow R \text{ is a linear operator then } f(x) = cx \text{ for some } c \in R. \) Assume
however that \( f : R \to R \) satisfies only the additive property, that is

\[
f(x + y) = f(x) + f(y)
\]

for every \( x, y \in R \).

- **Theorem 6** Let \( f : R \to R \) be such that for every \( x, y \in R \), \( f(x + y) = f(x) + f(y) \). If there exist \( a < b \) such that \( f \) is bounded on \([a, b]\) then \( f(x) = cx \) for some \( c \).

- Let \( \alpha, \beta \) be two irrational numbers such that \( \alpha / \beta \) is not in \( Q \). Then there is a function \( f : R \to R \) such that \( f(x + y) = f(x) + f(y) \) and \( f(\alpha) = 1, f(\beta) = 0 \) and \( f \) is not linear.
**Similarity**

Question: Let $L : V \to V$ be a linear operator and let $E, F$ be two ordered bases in $V$ with transition matrix $S$ (from $F$ to $E$). How does the matrix representation change as we switch from $E$ to $F$?

**Theorem 7** Let $E, F$ be two ordered bases for a vector space $V$ and let $L : V \to V$ has the matrix representation $A$ with respect to $E$. If $S$ is the transition matrix from $F$ to $E$ then the matrix representation $B$ of $L$ with respect to $F$ is

$$B = S^{-1}AS.$$ 

**Definition 3** Let $A, B$ be two $n \times n$ matrices. We say that $B$ is similar to $A$ if there is a nonsingular matrix $S$ such that $B =
$S^{-1}AS$.

**Note:** If $B$ is similar to $A$ then $A$ is similar to $B$. 