Chapters 6 and 7
Strong Induction

Aim: Prove that \( P(n) \) is true for every integer \( n \geq 1 \)

Method:

1. **Base Step**: Show that \( P(1) \) is true.

2. **Inductive Step**: Show that implication \( (P(1) \land P(2) \land \cdots \land P(n)) \rightarrow P(n + 1) \) is true for every integer \( n \).
**Theorem 1** Let \( a \) be an integer and let \( b \) be a positive integer. Then there exist integers \( q, r \) such that

\[
a = bq + r,
\]

\[0 \leq r < b.\]

**Theorem 2** Every integer \( n > 1 \) is either a prime or is a product of primes.

**Theorem 3 (Well-ordering principle)** Every non-empty set of natural numbers has a smallest element.
Irrationality of $\sqrt{n}$

- Show that $\sqrt{2}$ is irrational.
- Show that $\sqrt{3}$ is irrational.
**Denumerable sets**

**Definition 1** Two sets $A, B$ are equinumerous if there is a bijection $f : A \rightarrow B$.

$A \sim B$ if $A$ and $B$ are equinumerous.

**Example 1**

$\mathbb{Z}^+$ and $\mathbb{Z}$ are equinumerous.

Let $E$ be the set of even numbers. Then $E$ and $\mathbb{Z}^+$ are equinumerous.
Theorem 4  For any sets $A, B, C, D$.

- If $A \sim B$ and $C \sim D$, then $A \times C \sim B \times D$.

- If $A \sim B$, $C \sim D$, $A \cap C = \emptyset$, $B \cap D = \emptyset$, then $A \cup C \sim B \cup D$.

- $A \sim A$.

- If $A \sim B$, then $B \sim A$.

- If $A \sim B$ and $B \sim C$, then $A \sim C$. 
• A set $A$ is called **denumerable** if it is equinumerous with $\mathbb{Z}^+$.  

• A set $A$ is called **countable** if it is either finite or denumerable.  

• If $A$ is not countable then it is called **uncountable**.
Example 2  \( N \) is denumerable.

- \( Z \) is denumerable.

- The set of even integers is denumerable.

- The set \( \{5 + 1/n | n \in Z^+\} \) is denumerable.
Theorem 5  Every subset of a countable set is countable.

Theorem 6  • $\mathbb{Z}^+ \times \mathbb{Z}^+$ is denumerable.

• $\mathbb{Q}$ is denumerable.

Proof.

• Use $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, $f((a,b)) = 2^{a-1}(2b-1)$.

• Use $f : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$, $f(r) = (p,q)$ assuming $r = p/q$ and $p,q$ have no common factors.
**Theorem 7** If $A, B$ are denumerable then $A \cup B$ is denumerable and $A \times B$ is denumerable.

**Theorem 8** Let $\{A_i|i \in \mathbb{Z}^+\}$ be a family of pairwise disjoint sets such that $A_i$ is denumerable. Then $\bigcup_{i=1}^{\infty} A_i$ is denumerable.
Theorem 9  [Cantor’s Theorem] The set of real numbers is uncountable.

Georg Cantor, 1845-1918,
Proof. Can you enumerate all possible infinite strings of blue/red marbles?
Theorem 10  For every set $X$, $X$ and $P(X)$ are not equinumerous.
Proof. Show there is no surjection from $X$ to $P(X)$. By contradiction suppose $f : X \rightarrow P(X)$ is a surjection.

- Let $Y = \{x \in X | x \notin f(x)\}$.

- Let $y \in X$ be such that $f(y) = Y$.

- If $y \in f(y) = Y$ then by definition of $Y$, $y \notin Y$.

- If $y \notin f(y) = Y$ then by definition of $Y$, $y \in Y$. 