2. Basic Structures
2.1 Sets

**Definition 1** A set is an unordered collection of objects.

Important sets: \( \mathbb{N}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Q}, \mathbb{R} \).

**Definition 2** Objects in a set are called elements or members of the set. A set is said to contain its elements.

**Definition 3** Two sets are equal if they contain the same elements.

**Definition 4** The set \( A \) is a subset of \( B \) if and only if every element of \( A \) is also an element of \( B \). We will denote this by \( A \subseteq B \).
**Theorem 1** For any set $S$: (1) $\emptyset \subseteq S$, (2) $S \subseteq S$.

**Definition 5** Let $S$ be a set. If there are exactly $n$ distinct elements in $S$ then we say $S$ is a finite set and that the cardinality of $S$ is $n$. Cardinality of set $S$ is denoted by $|S|$.

A set is *infinite* if it is not finite.

**Definition 6** Given the set $S$, the power set $P(S)$ of $S$ is the set of all subsets of $S$.

**Fact 2** If $|S| = n$ then $|P(S)| = 2^n$. 
Definition 7 The ordered $n$-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has $a_1$ as its first element, $a_2$ as the second, $\ldots$, and $a_n$ as its $n$th element.

Definition 8 Let $A$, $B$ be sets. Then the Cartesian product of $A$ and $B$, $A \times B$ is the set

$$A \times B = \{(a, b) | a \in A \land b \in B\}.$$  

Definition 9 The Cartesian product of sets $A_1, \ldots, A_n$ is the set

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i\}.$$
2.2 Set Operations

Operations:

- **Union**: Let $A$, $B$ be sets. Then $A \cup B = \{ x | x \in A \lor x \in B \}$.

- **Intersection**: Let $A$, $B$ be sets. Then $A \cap B = \{ x | x \in A \land x \in B \}$.

- **Difference**: Let $A$, $B$ be sets. Then $A \setminus B = \{ x | x \in A \land x \neq B \}$.

- **Complement**: Let $U$ be the universal set. The complement of the set $A$, $\bar{A} = U \setminus A$. 
Fact 3  For any finite sets $A$ and $B$,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$ 

Two sets are called disjoint if their intersection is an empty set. Some set identities:

- $A \cup \emptyset = A$
- $A \cap \emptyset = \emptyset$
- $(\overline{A}) = A$
\begin{itemize}
\item $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
\item $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
\item $\overline{A \cup B} = \overline{A} \cap \overline{B}$
\item $\overline{A \cap B} = \overline{A} \cup \overline{B}$
\end{itemize}
Generalized unions and intersections:

Let $A_1, A_2, \ldots, A_n$ be sets.

$$\bigcup_{i=1}^{n} A_i = \{x | (x \in A_1) \lor (x \in A_2) \lor \ldots \lor (x \in A_n)\}$$

$$= \{x | \exists 1 \leq i \leq n x \in A_i\}$$

$$\bigcap_{i=1}^{n} A_i = \{x | (x \in A_1) \land (x \in A_2) \land \ldots \land (x \in A_n)\}$$

$$= \{x | \forall 1 \leq x \leq n x \in A_i\}$$
Example 1  Let \( A_i = \{i+5, i+6, \ldots, \} \). Find \( \bigcup_{i=1}^{n} A_i \) and \( \bigcap_{i=1}^{n} A_i \).

\[
\bigcup_{i=1}^{n} A_i = \{x| \exists_i x \in A_i\} = \{6, 7, 8, \ldots\}
\]

\[
\bigcap_{i=1}^{n} A_i = \{x| \forall_i x \in A_i\} = \{n + 5, n + 6, \ldots\}
\]
Example 2 Let $A_i = \{1, \ldots, 2i\}$. Find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$.

\[ \bigcup_{i=1}^{n} A_i = \{x|\exists_i x \in A_i\} = \{1, 2, \ldots, 2n\} \]

\[ \bigcap_{i=1}^{n} A_i = \{x|\forall_i x \in A_i\} = \{1, 2\} \]
Observations:

(1) $\forall 1 \leq j \leq n$

$$A_j \subseteq \bigcup_{i=1}^{n} A_i$$

(2) $\forall 1 \leq j \leq n$

$$\bigcap_{i=1}^{n} A_i \subseteq A_j$$

(3) If $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$ then

$$\bigcup_{i=1}^{n} A_i = A_n, \quad \bigcap_{i=1}^{n} A_i = A_1.$$
Computer representation of sets:
If a universal set $U = \{u_1, \ldots, u_n\}$ we can represent subsets of $U$ by binary strings of length $n$, where for set $A \subseteq U$ the $i$th but $a_i$ in the representation is equal to 1 if and only if $u_i \in A$. 
2.3 Functions

**Definition 10** Let $A$ and $B$ be sets. A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to every element of $A$. We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by $f$ to the element $a$. If $f$ is a function from $A$ to $B$, we write $f : A \to B$.

If $f : A \to B$, then $A$ is called a *domain* of $f$ and $B$ is called a *co-domain* of $f$. If $f(a) = b$ then $b$ is called an *image* of $a$ and $a$ is called a *pre-image* of $b$. The *range* of $f$ is the set of all images of elements in $A$.

**Definition 11** Let $f : A \to B$ be a function and let $S \subseteq A$. The
image of $S$ is defined as

$$f(S) = \{f(s) | s \in S\}.$$
Types of functions:

**Definition 12** A function \( f \) is said to be one-to-one (or injective) if and only if \( f(x) = f(y) \) implies that \( x = y \) for all \( x, y \) in the domain of \( f \).

\[
\forall x \forall y ((f(x) = f(y)) \rightarrow (x = y)).
\]

**Definition 13** A function \( f \) from \( A \) to \( B \) is called onto (or surjective) if for every \( b \) from \( B \) there is an \( a \) in \( A \) such that \( f(a) = b \).

\[
\forall b \in B \exists a \in A f(a) = b.
\]

**Definition 14** Function \( f \) is called a one-to-one correspondence (or a bijection) if it is both injective and surjective.
Injective function

Surjective Function

Bijective function
**Definition 15** A function $f$ whose domain and codomain are subsets of the set of real numbers is called

- **strictly increasing** if $f(x) < f(y)$ whenever $x < y$ and $x, y$ are in the domain of $f$.

- **strictly decreasing** if $f(x) > f(y)$ whenever $x < y$ and $x, y$ are in the domain of $f$. 
Operations:

- **Basic operations**: addition and multiplication.

- **Inverse function**: Let $f$ be a bijection from $A$ to $B$ then an inverse function is the function $f^{-1} : B \rightarrow A$ such that $f$ assigns to $b$ a unique element $a$ such that $f(a) = b$. A function that is not a bijection does not have an inverse.

- **Composition of functions**: Let $g$ be a function from $A$ to $B$ and let $f$ be a function from $B$ to $C$. Composition of $f$ and $g$ is the function $f \circ g : A \rightarrow C$ defined as $(f \circ g)(a) = f(g(a))$. 
• A function $id_A : A \to A$ such that for every $a \in A, id_A(a) = a$ is called an identity on $A$.

**Important functions:**

1. **Floor function:** $f(x) = \lfloor x \rfloor$ is equal to the largest integer less than or equal to $x$. 
2. **Ceiling function:** \( f(x) = \lceil x \rceil \) is equal to the smallest integer larger than or equal to \( x \).
Example 3  Graph the following functions:

1. \( f(x) = x - \lfloor x \rfloor \)

2. \( f(x) = \lceil x \rceil - \lfloor x \rfloor \)
2.4 Sequence and summations

Definition 16 A sequence is a function from the set of nonnegative (or positive) integers to some set $S$.

Two special sequences:

- arithmetic progression: $a, a + d, a + 2d, a + 3d, \ldots$

- geometric progression: $a, ar, ar^2, ar^3, \ldots$
Summations:

\[ \sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n. \]

Two special summations:

- The sum of the first \( n+1 \) terms of the arithmetic progression \( a, a + d, a + 2d, a + 3d, \ldots \) is
  \[ (n + 1)a + \frac{d(n + 1)n}{2}. \]

- The sum of the first \( n+1 \) terms of the geometric progression
when $r \neq 1$ and

\[
\frac{ar^{n+1} - a}{r - 1}
\]

when $r = 1$.  

when $r \neq 1$ and

\[
(n + 1)a
\]
Theorem 4

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \]

\[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \]

\[ \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}. \]