AN EXTENSION OF THE HAJNAL-SZEMERÉDI THEOREM TO DIRECTED GRAPHS

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Abstract. Hajnal and Szemerédi proved that every graph $G$ with $|G| = ks$ and $\delta(G) \geq k(s - 1)$ contains $k$ disjoint $s$-cliques; moreover this degree bound is optimal. We extend their theorem to directed graphs by showing that every directed graph $\overrightarrow{G}$ with $|\overrightarrow{G}| = ks$ and $\delta(\overrightarrow{G}) \geq 2k(s - 1) - 1$ contains $k$ disjoint transitive tournaments on $s$ vertices, where $\delta(\overrightarrow{G}) = \min_{v \in V(\overrightarrow{G})} d^-(v) + d^+(v)$. Our result implies the Hajnal-Szemerédi Theorem, and its degree bound is optimal. We also make some conjectures regarding even more general results for multigraphs and partitioning into other tournaments. One of these conjectures is supported by an asymptotic result.

1. Introduction

Let $G = (V, E)$ be a graph. An equitable $k$-coloring of $G$ is a proper $k$-coloring whose color classes differ in size by at most one. A factor of $G$ is a set $F$ of disjoint subgraphs whose union spans $G$. The subgraphs in a factor are called tiles. A factor $F$ is an $H$-factor if each of its tiles is a copy of $H$. If $|G| = sk$, then the color classes of an equitable $k$-coloring form a $K_s$-factor. Our story starts in 1963:

Theorem 1 (Corrádi & Hajnal 1963 [2]). Every graph $G$ with $|G| = n = 3k$ and $\delta(G) \geq \frac{2}{3}n$ has a $K_3$-factor.

The following generalization was conjectured by Erdős [7] in 1963, and proved seven years later:

Theorem 2 (Hajnal & Szemerédi 1970 [9]). Every graph $G$ with $|G| = n = ks$ and $\delta(G) \geq (1 - 1/s)n$ has a $K_s$-factor.

Since $|G| = \Delta(G) + \delta(\overrightarrow{G}) + 1$, Theorem 2 has the following complementary form, in which Hajnal and Szemerédi stated their proof of Erdős’ conjecture.

Theorem 3 (Hajnal & Szemerédi 1970 [9]). Every graph $G$ with $\Delta(G) \leq k - 1$ has an equitable $k$-coloring.

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The degree bounds in these theorems are easily seen to be tight. For example, \(G := K_{k,s} - E(K_{t+1})\) satisfies \(\delta(G) \geq (1 - 1/s)|G| - 1\) but has no \(K_s\)-factor. The original proof of Theorem 2 was quite involved, and only yielded an exponential time algorithm. Short proofs yielding polynomial time algorithms appear in [13, 16]; the following theorem provides a fast algorithm.

**Theorem 4** (Kierstead, Kostochka, Mydlarz & Szemerédi 2010 [15]). Every graph \(G\) on \(n\) vertices with \(\Delta(G) \leq k - 1\) can be equitably \(k\)-colored in \(O(kn^2)\) steps.

In this paper we consider extensions of Theorem 3 for simple digraphs—those having no loops and at most two edges \(xy, yx\) between any two vertices \(x, y\). The in- and out-degrees of a vertex \(v\) are denoted by \(d^-(v)\) and \(d^+(v)\); the total degree of \(v\) is the sum \(d(v) := d^-(v) + d^+(v)\). The minimum semi-degree of \(G\) is \(\delta^0(G) := \min \{\min \{d^+(v), d^-(v)\} : v \in V\}\) and the maximum semi-degree of \(G\) is \(\Delta^0(G) := \max \{\max \{d^+(v), d^-(v)\} : v \in V\}\). The minimum total degree of \(G\) is \(\delta(G) := \min \{d(v) : v \in V\}\) and the maximum total degree of \(G\) is \(\Delta(G) := \max \{d(v) : v \in V\}\). Among graphs on \(p\) vertices, let \(\overline{T}_p\) be the transitive tournament, \(\overline{C}_p\) be the directed cycle, and \(\overline{K}_p\) be the complete digraph with all possible edges in both directions. Let \(E^+(X,Y) = \{xy \in E(G) : x \in X \land y \in Y\}\).

The simplest way to extend Theorems 1 and 2 to digraphs is to replace minimum and maximum degree by (i) minimum and maximum total degree or (ii) minimum and maximum out-degree (equivalently in-degree); cliques are replaced by (transitive) tournaments and independent sets are replaced by acyclic sets—much more on the reasons for these choices later. The case \(s = 3\) is completely solved by the following two theorems:

**Theorem 5** (Wang 2000 [21]). Every digraph \(G\) with \(\delta(G) \geq \frac{3|G|-3}{2}\) has \(|G|/3\) disjoint copies of \(\overline{C}_3\). Moreover, for odd \(k \in \mathbb{Z}^+\), the digraph \(G := \overline{K}_{3k} - E^+(X,Y)\), where \(X\) and \(Y\) are disjoint and \(|X| = |Y| + 1 = \frac{3k+1}{2}\) satisfies \(\delta(G) = \frac{3|G|-5}{2}\), but does contain a \(\overline{C}_3\)-factor.

**Theorem 6** (Czygrinow, Kierstead & Molla 2012 [3]). Suppose \(G\) is a digraph with \(|G| = n = 3k\) and \(\delta(G) \geq 2 \cdot \frac{2}{3}n - 1\), and \(c \geq 0\) and \(t \geq 1\) are integers with \(c + t = k\). Then \(G\) has a factor consisting of \(c\) copies of \(\overline{C}_3\) and \(t\) copies of \(\overline{T}_3\). Moreover, for every \(k \in \mathbb{Z}^+\) the digraph \(G := \overline{K}_{3k} - E(\overline{K}_{k+1})\) satisfies \(|G| = n\) and \(\delta(G) = 2 \cdot \frac{2}{3}n - 2\), but has no factor whose tiles are tournaments on three vertices.

Theorem 3 gives an exact answer for cyclic 3-tournaments, and Theorem 6 with \(t = k\) gives an exact answer for transitive 3-tournaments. Moreover, it also shows that the same bound \(\delta(G) \geq 4k - 1\) also forces a factor with any combination of cyclic and transitive 3-tournaments, except for \(k\) cyclic tournaments.

The extremal example for Theorem 6 is the natural extension of the extremal example for Theorem 2 more generally the digraph \(G := \overline{K}_{sk} - E(\overline{K}_{k+1})\) satisfies \(|G| = n = sk\) and \(\delta(G) \geq 2(1 - 1/s)n - 2\), but does not have a factor whose tiles are \(s\)-tournaments.
The extremal example for Theorem 5 seems to be an accident of small numbers—it works because it is not strongly connected. If \( s \geq 4 \) and \( \delta(G) \geq 2(1-1/s)n-2 \), then \( G \) is strongly connected.

Our main result is:

**Theorem 7.** Every digraph \( G \) with \( |G| = n = sk \) and \( \delta(G) \geq 2(1-1/s)n-1 \) has a \( \overrightarrow{T}_s \)-factor.

We prove Theorem 7 in its following stronger complementary form by extending ideas developed in [13, 14, 15, 16]. An equitable acyclic coloring of a digraph is a coloring whose classes induce acyclic subgraphs (subgraphs with no directed cycles, including 2-cycles), and differ in size by at most one.

**Theorem 8.** Every digraph \( G \) with \( \Delta(G) \leq 2k-1 \) has an equitable acyclic \( k \)-coloring.

To see that Theorem 8 implies Theorem 7 consider a digraph \( G \) with \( |G| = n = sk \) and \( \delta(G) \geq 2(1-1/s)n-1 \). Its complement \( H \) satisfies \( \Delta(H) \leq 2n-2-(2(1-1/s)n-1) \leq 2k-1 \). By Theorem 8, \( H \) has an equitable acyclic \( k \)-coloring. Since each color class is acyclic it can be embedded in a transitive \( s \)-tournament, whose complement is another transitive tournament contained in \( G \). Thus the tiles in \( G \) induced by the color classes of \( H \) contain transitive \( s \)-tournaments.

Even though a color class of an acyclic coloring may contain many edges, Theorem 8 is stronger than Theorem 3. To see this, let \( G \) be a graph with \( \Delta(G) \leq k-1 \) and let \( D \) be the graph obtained by replacing edge \( uv \) of \( G \) with two directed edges \( uv \) and \( vu \). Since \( \Delta(D) \leq 2k-2 \), we may apply Theorem 8 to obtain an equitable acyclic \( k \)-coloring. Note that there are no edges in any color class since if \( uv \) was in a color class, then \( vu \) would be as well giving us a directed 2-cycle. Thus the equitable acyclic \( k \)-coloring of \( D \) induces an equitable coloring of \( G \).

The following two statements are neither implied by Theorem 7 nor Theorem 8 nor do they imply Theorem 7 or Theorem 8. However our proof can be slightly modified to give these results as well:

**Theorem 9.**

(i) Every digraph \( G \) with \( |G| = n = sk \) and \( \delta^+(G) \geq (1-1/s)n \) has a \( \overrightarrow{T}_s \)-factor.

(ii) Every digraph \( G \) with \( \Delta^+(G) \leq k-1 \) has an equitable acyclic \( k \)-coloring.

The paper is organized as follows. In the remainder of this section we introduce some more notation. In Section 2 we prove Theorem 8. In Section 3 we introduce further conjectures concerning tiling with nontransitive tournaments, and generalizations to multigraphs. In Section 4 we support one of these, Conjecture 18 by proving an asymptotic version.

1.1. **Notation.** For a digraph \( G = (V,E) \) set \( |G| = |V| \) and \( ||G|| = |E| \). Also, set \( E^+(X,Y) = E^-(Y,X) = \{ xy \in E : x \in X \land y \in Y \} \), and \( E(X,Y) = E^+(X,Y) \cup E^-(X,Y) \). Set \( ||X,Y|| = |E(X,Y)|, ||X,Y||^+ = |E^+(X,Y)| \) and \( ||X,Y||^- = |E^-(X,Y)| \). An edge \( e \) is heavy if it is contained in a 2-cycle; otherwise it is light. Let \( ||X,Y||^h \) denote the number of 2-cycles contained in \( E(X,Y) \). Then \( 2||X,Y||^h \) is the number of heavy edges in \( E(X,Y) \).
Let \( \|X, Y\| \) denote the number of light edges in \( E(X, Y) \). We shorten \( E(\{x\}, Y) \) to \( E(x, Y) \) and \( E(X, V) \) to \( E(X) \), etc.

## 2. Main Result

In this section we prove Theorem \(^8\) Our proof is based on the proof of Theorem \(^4\) Although we do not go into the details, it also provides an \( O(kn^2) \) algorithm. Otherwise, our proof could be slightly simplified by avoiding the use of \( B' \).

For simplicity, we shorten equitable acyclic to good.

**Proof of Theorem \(^8\)** We may assume \( |G| = sk \), where \( s \in \mathbb{N} \): If \( |G| = sk - p \), where \( 1 \leq p < k \), then let \( G' \) be the disjoint union of \( G \) and \( \overline{K}_p \). Then \( |G'| \) is divisible by \( k \), and \( \Delta(G') \leq 2k - 1 \), any good \( k \)-coloring of \( G' \) induces a good \( k \)-coloring of \( G \).

Argue by induction on \( \|G\| \). The base step \( \|G\| = 0 \) is trivial; so suppose \( u \) is a non-isolated vertex. Set \( G' := G - E(u) \). By induction, \( G' \) has a good \( k \)-coloring \( f \). We are done unless some color class \( U \) of \( f \) contains a cycle \( C \) with \( u \in C \). Since \( \Delta(G) \leq 2k - 1 \), for some class \( W \) either \( \|u, W\|^- = 0 \) or \( \|u, W\|^+ = 0 \). Moving \( u \) from \( U \) to \( W \) yields an acyclic \( k \)-coloring of \( G \) with all classes of size \( s \), except for one small class \( U - u \) of size \( s - 1 \) and one large class \( W + u \) of size \( s + 1 \). Such a coloring is called a nearly equitable acyclic \( k \)-coloring. We shorten this to useful \( k \)-coloring.

For a useful \( k \)-coloring \( f \), let \( V^- := V^- (f) \) be the small class and \( V^+ := V^+ (f) \) be the large class of \( f \), and define an auxiliary digraph \( H := H(f) \), whose vertices are the color classes, so that \( UW \) is a directed edge if and only if \( U \neq W \) and \( W + y \) is acyclic for some \( y \in U \). Such a \( y \) is called a witness for \( UW \). If \( W + y \) contains a directed cycle \( C \), then we say that \( y \) is blocked in \( W \) by \( C \). If \( y \) is blocked in \( W \), then

\[
\|W, y\| \geq 2.
\]

Let \( A \) be the set of classes that can reach \( V^- \) in \( H \), \( B \) be the set of classes not in \( A \), and \( B' \) be the set of classes that can be reached from \( V^+ \). Call a class \( W \in A \) terminal, if every \( U \in A - W \) can reach \( V^- \) in \( H - W \); so \( V^- \) is terminal if and only if \( A = \{V^-\} \). Let \( A' \) be the set of terminal classes. A class \( A \) in \( H \) with maximum distance to \( V^- \) in \( H \) is terminal; so \( A' \neq \emptyset \). For any \( W \in V(H) \) and any \( x \in W \) we say \( x \) is \( q \)-movable if it witnesses exactly \( q \) edges in \( E^+(W, A) \). If \( x \) is \( q \)-movable for \( q \geq 1 \), call \( x \) movable. Set \( a := |A|, a' := |A'|, b := |B|, b' := |B'|, A := \bigcup A, A' := \bigcup A', B := \bigcup B \) and \( B' := \bigcup B' \). An edge \( e \in E(A, B) \) is called a crossing edge; denote its ends by \( e_A \) and \( e_B \), where \( e_A \in A \).

**Claim 1.** If \( V^+ \in A \), then \( G \) has a good \( k \)-coloring.

**Proof.** Let \( P = V_1 \ldots V_k \) be a \( V^+, V^- \)-path in \( H \). Moving witnesses \( y_j \) of \( V_j V_{j+1} \) to \( V_{j+1} \) for all \( j \) yields a good \( k \)-coloring of \( G \).

Establishing the next lemma completes the proof; notice the weaker degree condition.

**Lemma 10.** A digraph \( G \) has a good \( k \)-coloring provided it has a useful \( k \)-coloring \( f \) with

\[
d(v) \leq 2(k - 1) (= 2a + 2b - 1) \text{ for every vertex } v \in A' \cup B. \]
Proof. Arguing by induction on \( k \), assume \( G \) does not have a good \( k \)-coloring.

A crossing edge \( e \) with \( e_A \in W \in \mathcal{A} \) is vital if \( G[W + e_B] \) contains a directed cycle \( C \) with \( e \in E(C) \). In particular if \( xy \) is a crossing edge with \( \|x,y\| = 2 \), then both \( xy \) and \( yx \) are vital. For sets \( S \subseteq A \) and \( T \subseteq B \) denote the number of vital edges in \( E(S,T), E^-(S,T), \) and \( E^+(S,T) \) by \( \nu(S,T), \nu^-(S,T), \) and \( \nu^+(S,T) \), respectively. If \( S = \{x\} \) or \( T = \{y\} \), we drop the braces. Every \( y \in B \) is blocked in \( W \); so \( \nu^+(W,y), \nu^-(W,y) \geq 1 \) and

\[
\nu(W,y) \geq 2. \tag{2.3}
\]

**Claim 2.**  For any \( x \in W \in \mathcal{A}' \), if \( x \) is \( q \)-movable, then

\[
\text{(a) } \|x, B\| \leq 2(b + q) + 1 - \|x, W\| \quad \text{and} \quad \text{(b) } \nu(x, B) \leq 2(b + q).
\]

**Proof.** (a) There are \((a - 1) - q\) classes in \( A \setminus W \) in which \( x \) is blocked. So \((2.1)\) gives that \( \|x, A \setminus W\| \geq 2a - 2q - 2 \). With \((2.2)\), this implies

\[
\|x, B\| \leq 2a + 2b - 1 - \|x, A \setminus W\| - \|x, W\| \leq 2(b + q) + 1 - \|x, W\|.
\]

(b) By (a), \( \nu(x, B) \leq 2(b + q) + 1 - \|x, W\| \). So the desired inequality holds if \( \|x, W\| \geq 1 \) or \( \nu(x, B) \) is even. If \( \|x, W\| = 0 \), then every vital edge incident to \( x \) must be heavy. This implies that \( \nu(x, B) \) is even. \(\square\)

**Claim 3.** \( V^- \) is not terminal.

**Proof.** If \( V^- \) is terminal, then \( \mathcal{A} = \{V^-\} \) and \( a = 1 \); thus there are no movable vertices. Claim \((2.1)\) implies \( \nu(u, B) \leq 2b \) for all \( u \in A \) and \((2.3)\) implies \( \nu(A, w) \geq 2 \) for all \( w \in B \). This yields the contradiction

\[
2(bs + 1) = 2|B| \leq \nu(A, B) \leq 2b(s - 1).
\]

\(\square\)

Using Claim \((1)\) and Claim \((3)\), \( V^+ \in B \) and \( A \neq \mathcal{A}' \); thus

\[
|A| = as - 1, |A'| = a's, |B| = bs + 1, \quad \text{and} \quad |B'| = b's + 1. \tag{2.4}
\]

The next claim provides a key relationship between vertices in \( A' \) and vertices in \( B \).

**Claim 4.** For all \( x \in W \in \mathcal{A}' \), and \( y \in B \):

(a) if \( G[W - x + y] \) is acyclic, then \( x \) is not movable; and
(b) there is no \( y' \in B' - y \) such that \( G[W - x + y + y'] \) is acyclic.

**Proof.** By Claim \((1)\) and Claim \((3)\), \( W \notin \{V^-, V^+\} \). Suppose there exists \( y \in B \) such that \( G[W - x + y] \) is acyclic. If there exists \( y' \in B' - y \) such that \( G[W - x + y + y'] \) is acyclic, put \( y_1 := y', y_2 := y \) and \( Y := \{y_1, y_2\} \); else put \( y_1 := y \) and \( Y := \{y_1\} \). Since \( W \in \mathcal{A} \), it contains a movable vertex. If \( x \) is movable put \( x' := x; \) else let \( x' \in W \) be any movable vertex; say \( x' \) witnesses \( WU \), where \( U \in \mathcal{A} \). Let \( X := \{x', x\} \) and \( W' := W \setminus X + y_1 \).

Moving \( x' \) to \( U \) and switching witnesses along a \( U,V^-\)-path in \( \mathcal{H} - W \) yields a good \((a - 1)\)-coloring \( f_1 \) of \( G_1 := G[A \setminus W + x'] \). Also \( f \) induces a \( b \)-coloring \( f_2 \) of \( G_2 := G[B - y_1] \). It is good if \( y_1 \in V^+ \); else it is useful. Since every \( v \in B - y_1 \) is blocked in every color class
in $\mathcal{A}(f)$, (2.1) and (2.2) imply $\Delta(G_2) \leq 2k - 1 - 2a = 2b - 1$. By induction, there is a good $b$-coloring $g_2$ of $G_2$. (For algorithmic considerations, note that if $y_1 \in B'$, as when $|Y| \geq 2$, then $g_2$ is immediately constructible from $f_2$ using Claim 1.)

If $|X| = 1$, then $|W'| = s$. So $g_1$, $W'$ and $g_2$ form a good $k$-coloring of $G$. This completes the proof of (a) (see Figure 2.1). To prove (b), suppose $|X| = |Y| = 2$. It suffices to show that $G_3 := G[(B - y_1) + (W' + x)]$ has a good $(b + 1)$-coloring. By the case, $x$ is blocked in every class of $\mathcal{A} \setminus W$; so $\|x, A \setminus W\| \geq 2a - 2$ by (2.1). Thus $Z + x$ is acyclic for some class $Z \in B + W'$. So $G_3$ has a useful $(b + 1)$-coloring $f_3$ with $V^-(f_3) = W'$ and $V^+(f_3) = Z + x$, or $Z = W'$ and $f_3$ is already good. Since $W' + y_2$ is acyclic, $W' \notin \mathcal{A}(f_3) \cup \mathcal{B}(f_3)$. By the definitions of $x$ and $B(f)$, every $v \in V(G_3) \setminus W'$ is blocked in every color class in $\mathcal{A}(f) - W$. Thus, by (2.1) and (2.2), $\|v, V(G_3)\| \leq 2(b + 1) - 1$. So, by induction, there exists a good $(b + 1)$-coloring $g_3$ of $G_3$ (see Figure 2.2).

A crossing edge $e \in E(W, B)$ is lonely if it is vital and either (i) $e \in E^-(W, B)$ and $\nu^-(W, e_B) = 1$ or (ii) $e \in E^+(W, B)$ and $\nu^+(W, e_B) = 1$. If (i), then $e$ is in-lonely; if (ii), then $e$ is out-lonely. If $e$ is lonely, then $G[W - e_A + e_B]$ is acyclic. For sets $S \subseteq A$ and $T \subseteq B$ denote the number of lonely, in-lonely and out-lonely edges in $E(S, T)$ by $\lambda(S, T)$, $\lambda^-(S, T)$ and $\lambda^+(S, T)$, respectively; drop braces for singletons. If $y \in B$, then $y$ is blocked in $W$. So

$$
\nu(W, y) + \lambda(W, y) \geq 4.
$$

(2.5)

**Claim 5.** $a' > b$.

**Proof.** Assume $a' \leq b$. Order $\mathcal{A}$ as $X_1 := V^-, X_2, \ldots, X_a$ so that for all $j > 1$ there exists $i < j$ with $X_iX_j \in E(\mathcal{H})$, and subject to this, order $\mathcal{A}$ so that $l$ is maximum, where $l$ is the largest index of a non-terminal class. Set $W := X_a$.

![Figure 2.1](image-url) After moving $x$ and $y_1$ as indicated, switching witness along a $U, V^-$-path in $\mathcal{H} - W$ creates a good $a$-coloring of $G[A + y_1]$. By induction, there is a good $b$-coloring of $G[B - y_1]$. 

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**Figure 2.1.** After moving $x$ and $y_1$ as indicated, switching witness along a $U, V^-$-path in $\mathcal{H} - W$ creates a good $a$-coloring of $G[A + y_1]$. By induction, there is a good $b$-coloring of $G[B - y_1]$. 

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![Diagram](image-url)
Figure 2.2. After moving $x$ and $y_1$ as indicated, switching witnesses (one of which is $x'$) creates a good $(a - 1)$-coloring of $G[A - W + x']$ and a good $b$-coloring $g_2$ of $G[B - y_1]$. Placing $x$ in a color class of $g_2$ gives a useful $(b + 1)$-coloring of $G[B + W - x']$ with small class $W' := W - x - x' + y_1$. By induction, there is a good $(b + 1)$-coloring of $G[B + W - x']$ because $G[W' + y_2]$ is acyclic.

The deletion of any non-terminal class leaves some class which can no longer reach $V^-$ in $H$; thus $l < a$, i.e., $W$ is terminal. Also $N^+_H(W) \subseteq \mathcal{A}' + X_l$, since otherwise we could increase the index $l$ by moving $W$ in front of $X_l$. So if $x \in W$ is $q$-movable, then

$$ q \leq a'. \tag{2.6} $$

If $\lambda(x, B) \geq 1$, then there exists $y \in B$ such that $\nu(x, y) = 1$ or $\nu(x, y) = 1$. In either case, $W - x + y$ is acyclic. Therefore, Claim 4(a) implies $q = 0$; since every lonely edge is vital, this and Claim 2(b) imply, $\lambda(x, B) \leq \nu(x, B) \leq 2b$. If $\lambda(x, B) = 0$, then Claim 2(b) and (2.6) gives $\nu(x, B) \leq 2(b + a') \leq 4b$. Regardless, $\lambda(x, B) + \nu(x, B) \leq 4b$. So

$$ \lambda(W, B) + \nu(W, B) = \sum_{x \in W} \lambda(x, B) + \nu(x, B) \leq 4b|W| \leq 4bs. $$

This is a contradiction, since (2.5) and (2.4) imply

$$ \lambda(W, B) + \nu(W, B) = \sum_{y \in B} \lambda(W, y) + \nu(W, y) \geq 4|B| > 4bs. \tag{2.7} $$

A crossing edge $e \in E(W, B)$ is solo if either (i) $e \in E^-(W, B)$ and $\|W, e_B\|^- = 1$ or (ii) $e \in E^+(W, B)$ and $\|W, e_B\|^+ = 1$. If (i), then $e$ is in-solo; if (ii), then $e$ is out-solo. For sets $S \subseteq A$ and $T \subseteq B$ denote the number of solo, in-solo and out-solo edges in $E(S, T)$ by $\sigma(S, T)$, $\sigma^-(S, T)$ and $\sigma^+(S, T)$, respectively; drop braces for singletons. If $y \in B$, then $y$ is blocked in $W$. So

$$ \|W, y\| + \sigma(W, y) \geq 4. \tag{2.7} $$
Every \( y \in B' \) is blocked in every color class in \( \mathcal{A} \cup (\mathcal{B} \setminus \mathcal{B}') \). So (2.1) and (2.2) give
\[
\|A', y\| \leq 2a + 2b - 1 - \|A \setminus A', y\| - \|B \setminus B', y\| - \|y, B'\| \leq 2a' + 2b' - 1 - \|y, B'\|. \tag{2.8}
\]
Using (2.7) and (2.8) we have
\[
\sigma(A', y) \geq \sum_{w \in A'} (4 - \|W, y\|) = 4a' - \|A', y\| \geq 2a' - 2b' + \|y, B'\| + 1
= 2(a' - b') + 2\|y, B'\|^h + \|y, B'\|^l + 1. \tag{2.9}
\]
Choose a maximal set \( I \) subject to \( V^+ \subseteq I \subseteq B' \) and \( G[I] \) contains no 2-cycle. Let
\[
J := \{ y \in I : \sigma(A', y) = 2(a' - b') + 2\|y, B'\|^h + 1 \}.
\]
Note that, by (2.9), the vertices in \( J \) have the minimum possible number of solo-neighbors in \( A' \) and additionally are incident with no light edges in \( B' \).

**Claim 6.** Every \( x \in A' \) satisfies \( \sigma(x, I) \leq 2 \). Furthermore, if there are distinct \( y_1, y_2 \in I \) such that \( \sigma(x, y_1), \sigma(x, y_2) \geq 1 \), then \( \{y_1, y_2\} \subseteq I \setminus J \).

**Proof.** Suppose \( \sigma(x, I) \geq 3 \) for some \( x \in W \in A' \). By Claim 3 \( W \neq V^- \). There exist distinct \( y_1, y_2 \in I \) such that either \( \sigma^+(x, \{y_1, y_2\}) = 2 \) or \( \sigma^-(x, \{y_1, y_2\}) = 2 \). Suppose \( \sigma^+(x, \{y_1, y_2\}) = 2 \). Then \( \|y_i, W - x + y_i\|^+ = 0 \) for each \( i \in [2] \). The choice of \( I \) implies \( \|y_1, y_2\| \leq 1 \). So there exists \( i \in [2] \) with \( \|y_i, W - x + y_1 + y_2\|^+ = 0 \). Thus \( G[W - x + y_1 + y_2] \) is acyclic, contradicting Claim 4(b).

Now suppose there exist distinct \( y_1 \in J \) and \( y_2 \in I \) with \( \sigma(x, y_1), \sigma(x, y_2) \geq 1 \). By the definition of \( J \) and (2.9), \( \|y_1, B'\|^l = 0 \). Therefore, by the definition of \( I \), \( \|y_1, I\| = 0 \) and in particular \( \|y_1, y_2\| = 0 \). So again \( G[W - x + y_1 + y_2] \) is acyclic, contradicting Claim 4(b).

The maximality of \( I \) implies that for all \( y \in B \setminus I \) there exists \( v \in I \) with \( \|y, v\| = 2 \). Therefore,
\[
\sum_{y \in I}(2\|y, B'\|^h + 2) = 2\|B' \setminus I, I\|^h + 2|I| \geq 2|B' \setminus I| + 2|I| = 2|B'|. \tag{2.10}
\]
Also, by (2.9) and the definition of \( J \), for every \( y \in I \setminus J \),
\[
\sigma(A', y) \geq 2(a' - b') + 2\|y, B'\|^h + 2. \tag{2.11}
\]
Therefore, by (2.11), (2.10), (2.4), Claim 5 and the fact that \( |I| \geq |V^+| > s \),
\[
\sigma(A', I) + |J| = \sum_{y \in I \setminus J} \sigma(A', y) + \sum_{y \in J} (\sigma(A', y) + 1)
\geq \sum_{y \in I}(2(a' - b') + 2\|y, B'\|^h + 2) > 2s(a' - b') + 2|B'| > 2|A'|
\sigma(A', I) > 2|A'| - |J|. \tag{2.12}
\]
Claim 6 only gives \( \sigma(A', I) \leq 2|A'| \), so we have not reached a contradiction yet. However, we will be saved by the fact that every vertex in \( J \) forces at least one fewer solo edge between
Conjecture 12. For every $A' \in A$ and $I$. Formally, let $A'_1 := \{x \in A' : \sigma(x, I) \leq 1\}$ and note that we can now write
\[ \sigma(A', I) \leq 2|A'| - |A'_1|. \] 

Claim 7. $|A'_1| \geq |J|$

Proof. For any $y \in J$, by the definition of $J$, $\sigma(A', y)$ is odd. This implies that there exists $x \in A'$ such that $\sigma(x, y) = 1$. By Claim 6, $\sigma(x, y') = 0$ for all $y' \in I - y$. Therefore $x \in A'_1$. □

Finally by (2.12), (2.13), and Claim 7
\[ 2|A'| - |J| < \sigma(A', I) \leq 2|A'| - |A'_1| \leq 2|A'| - |J|, \]
a contradiction. This completes the proof of Lemma 10. □

Applying Lemma 10 to the useful $k$-coloring $f$ completes the proof of Theorem 8. □

3. Conjectures

Removing the orientation from the edges of a directed graph $D$ leaves a loopless multigraph $M$ such that every edge has multiplicity at most 2. Call such a multigraph standard, and say that $M(D)$ is the multigraph underlying $D$. For a fixed standard multigraph $M$, let $H(M)$ and $L(M)$ be the graphs on $V(M)$ containing the edges of $M$ with multiplicity 2 and 1 respectively, and put $G(M) := H(M) \cup L(M)$.

Conjecture 11. Every standard multigraph $M$ with $\Delta(M) \leq 2k - 1$ has an equitable acyclic $k$-coloring.

We normally state Conjecture 11 in the following complimentary form. The complete standard multigraph $K^2_s$ on $s$ vertices is defined so that $H(K^2_s) = K_s$. The complement of a standard multigraph $M$ is $\overline{M} := K^2_s - M$, where $\mu_{\overline{M}}(xy) = 2 - \mu_M(xy)$. The complement of an acyclic standard multigraph on $s$-vertices is called a full $s$-clique.

Conjecture 12. For every $s, k \in \mathbb{N}$, if $M$ is a standard multigraph on $sk$ vertices and $\delta(M) \geq 2(s-1)k - 1$, then $M$ contains $k$ disjoint full $s$-cliques.

For the case $s = 3$, Conjecture 12 is a corollary of the main theorem in [3]. We also make the following conjecture based on the work in [3].

Conjecture 13. If $D$ is a strongly 2-connected digraph on $3k$ vertices such that $\delta(D) \geq 4k - 1$, then $D$ has a cyclic triangle factor.

By slightly modifying the example from Theorem 5, we see that this is best possible: For any $p \in \mathbb{N}^+$, let $k := 2p + 1$ and $D' := \overline{K}_{3k} - E^+(X,Y)$, where $X$ and $Y$ are disjoint and $|X| = 3p + 2, |Y| = 3p + 1$. To form the example $D$, reverse the orientation of the edges...
incident to some \( v \in Y \). \( D \) is strongly connected and \( \delta(D) = 9p + 2 = \frac{3|D| - 3}{2} - 1 \) but \( D \) has no cyclic triangle factor, because all cyclic triangles in \( D \) that intersect \( X \) and \( Y \) contain \( v \) and a vertex in \( Y - v \).

Let \( K \) be an \( s \)-clique and let \( D \) be the set of all simple digraphs \( D \) such that \( K = M(D) \) (equivalently the set of all simple digraphs obtained by orienting the edges of \( K \)); we say \( K \) is \textit{universal} if for all \( D \in \mathcal{D} \), \( D \) contains every tournament on \( s \) vertices. For example, the 3-clique \( Q = K_2^3 - e \) on 3 vertices with 5 edges is universal—every orientation of \( Q \) contains both \( \vec{C}_3 \) and \( \vec{K}_3 \). Our goal is to factor standard multigraphs into universal tiles.

Note that \( K \) is universal if and only if for every tournament \( T \) on \( s \) vertices and every orientation \( D \) of \( L(K) \) there is an embedding of \( D \) into \( T \) (after embedding \( D \) into \( T \), every other edge of \( T \) corresponds to a heavy edge of \( K \)). The following Theorem of Havet and Thomassé and famous conjecture of Sumner, which has been proved for large values of \( n \) [18], allow us to concisely say which cliques are universal.

**Theorem 14** (Havet & Thomassé 2000 [10]). Every tournament \( T \) on \( n \) vertices contains every oriented path \( P \) on \( n \) vertices except when \( P \) is the anti-directed path and \( n \in \{3, 5, 7\} \).

**Conjecture 15** (Sumner 1971). Every orientation of every tree on \( n \) vertices is a subgraph of every tournament on \( 2n - 2 \) vertices.

With Theorem 14 we can state Conjecture 15 in a form that is more useful for our goal.

**Conjecture 16.** Let \( T \) be a tournament on \( n \) vertices and \( F \) be a forest on at most \( n \) vertices with \( c \) non-trivial components. If \( F \) has at most \( n/2 + c - 1 \) edges, then \( T \) contains every orientation of \( F \).

**Proposition 17.** Theorem 14 and Conjecture 15 imply Conjecture 16.

**Proof.** Assume Conjecture 15 is true. Let \( D \) be an orientation of \( F \). We will argue by induction on \( c \). Let \( D_1 \) be the largest component in \( D \), \( D_2 := D - D_1 \) and \( m_i := \|D_i\| \) for \( i \in \{1, 2\} \). We can assume that \( m_1 \geq 3 \). Indeed, if \( m_1 \leq 2 \), then \( F \) is a collection of disjoint paths each on at most 3 vertices. Since \( \|F\| \leq 1 \) when \( n = 3 \), Theorem 14 implies that there is an embedding of \( D \) into \( T \).

Because there are \( c - 1 \) non-trivial components in \( D_2 \), \( m_2 \geq c - 1 \). Therefore, \( m_1 \leq n/2 \) and, since \( D_1 \) is a tree, \( 2|D_1| - 2 \leq 2(n/2 + 1) - 2 = n \). Conjecture 15 then implies that there is an embedding \( \phi \) of \( D_1 \) into \( T \). Note that this handles the case when \( c = 1 \).

Let \( T_2 := T - \phi(V(D_1)) \). Since \( m_1 \geq 3 \) we have that \( m_1 \geq (m_1 + 1)/2 + 1 \). Therefore, \( \|D_2\| \leq n/2 + c - 1 - m_1 \leq (n - m_1 - 1)/2 + (c - 1) - 1 \). Since \( |T_2| = n - m_1 - 1 \), there is an embedding of \( D_2 \) into \( T_2 \) by induction.

So if Sumner’s conjecture is true, then universal \( s \)-cliques are those whose light edges induce a forest with \( c \) non-trivial components and at most \( s/2 + c - 1 \) edges. In light of this, we conjecture the following.

**Conjecture 18.** For every \( s \geq 4 \) and \( k \in \mathbb{N} \), if \( M \) is a standard multigraph on \( sk \) vertices with \( \delta(M) \geq 2(s - 1)k - 1 \), then \( M \) can be tiled with \( k \) disjoint universal \( s \)-cliques.
In the following section we support Conjecture 18 with Theorems 20 and 23. Combining Theorem 6, Conjectures 13, 15 and 18 with Proposition 17 we have the following.

**Conjecture 19.** For any $s,k \in \mathbb{N}$, if $D$ is a strongly 2-connected digraph on $sk$ vertices and $\delta(D) \geq 2(s-1)k-1$, then $D$ contains any combination of $k$ disjoint tournaments on $s$ vertices.

4. An Asymptotic Result

Let $K$ be a full clique on at most $s$ vertices. It is fit if $\|K\|^l \leq \max\{0,|K|-s/2\}$. It is a near matching if either $\|v,K\|^l \leq 1$ for every vertex $v \in K$; or $|K| = s$, $\|v,K\|^l \leq 2$ for every vertex $v \in K$ and $\|v,K\|^l = 2$ for at most one vertex $v \in K$. It is acceptable if it is fit or a near matching.

In this section we prove the following theorem.

**Theorem 20.** For all $s \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n_0$ such that if $M$ is a standard multigraph on $n \geq n_0$ vertices, where $n$ is divisible by $s$, then the following holds. If $\delta(M) \geq 2(1-1/s)n + \varepsilon n$, then there exists a perfect tiling of $M$ with acceptable $s$-cliques.

If $K$ is acceptable and $|K| \geq 4$ then $L(K)$ is a forest with at most $|K|/2 + c - 1$ edges where $c$ is the number of components of $L(K)$. Therefore, with Proposition 17 and the fact that Conjecture 15 is true for large trees [13], we have the following corollary.

**Corollary 21.** There exists $s_0$ such that for any $s \geq s_0$ and any $\varepsilon > 0$ there exists $n_0$ such that if $D$ is a directed graph on $n \geq n_0$ vertices, where $n$ is divisible by $s$, the following holds. If $\delta(D) \geq 2(1-1/s)n + \varepsilon n$, then $D$ can be partitioned into tiles of order $s$ such that each tile contains every tournament on $s$ vertices.

First we show with Theorem 23 that for fixed $s$ we can tile all but at most a constant number of vertices of $M$ with universal $s$-cliques.

The following is a key step in the proof.

**Lemma 22.** Let $1 \leq t \leq s-1$ and suppose $M$ is a standard multigraph. If $X_1$ and $X_2$ are fit $t$-cliques, $Y$ is a fit $s$-clique, and $\|X_i,Y\| \geq 2(s-1)t + 2 - i$ for $i \in \{2\}$, then $M[X_1 \cup X_2 \cup Y]$ contains two disjoint fit cliques with orders $t+1$ and $s$ respectively.

**Proof.** Put $Y^c_i := \{y \in Y : \|X_i,y\| = 2t - c\}$ and choose $x_1 \in X_1$ with $\|x_1,Y\|^l \leq 1$.

Assume there exists $y \in Y^0_1 \cup Y^0_2$ such that $\|y,Y\|^l \geq 1$. If $y \in Y^0_2$, then $Y - y + x_1$ and $X_2 + y$ are fit. If $y \in Y^0_1 \setminus Y^0_2$, then $X_1 + y$ is fit and $\|X_2,Y - y\| \geq 2(s-1)t - (2t-1) = 2(s-2)t + 1$ so there exists $x_2 \in X$ such that $\|x_2,Y - y\|^l \leq 1$ and $Y - y + x_2$ is fit.

So we can assume $\|Y^0_1 \cup Y^0_2,Y\|^l = 0$. Since $|Y^0_1| + (2t-1)s \geq 2|Y^0_1| + |Y^1_1| + (2t-2) \geq \|X_i,Y\| \geq 2(s-1)t + 2 - i$, we have

\[ (a) \, |Y^0_1| \geq s - 2t + 2 - i \quad \text{and} \quad (b) \, |Y^0_1| + \frac{1}{2}|Y^1_1| \geq s - t + 1 - \frac{i}{2}. \quad (4.1) \]
By (4.1), if \( t < s/2 \) there exists \( y_2 \in Y^0 \) and if \( t \geq s/2 \) there exists \( y_2 \in Y^0 \cup Y^1 \). Note that in either case \( X_2 + y_2 \) is fit. As \( Y \) is full, \( \alpha(L[Y^1]) \geq \frac{1}{2}|Y_1| \). So by (4.1(b)) there exists \( I_1 \subseteq Y_1 \) such that \( \|I_1 \cup Y^0\|^t = 0 \) and \( |I_1 \cup Y^0| \geq s - t + 1 \). Therefore we can select \( Z_1 \subseteq I_1 \cup Y^0 - y_2 \) such that \( |Z_1| = s - t \) and, by (4.1(a)) \( |Z_1 \cap Y^0| \geq s - 2t \). \( X_1 \cup Z_1 \) is full and, because \( \|Z_1 \cap Y^1\|^t = |Z_1 \cap Y^1| \leq \min\{t, s - t\} \leq s/2 \), \( X_1 \cup Z_1 \) is fit.

\[\square\]

**Theorem 23.** Let \( s \geq 2 \) and let \( M \) be a standard multigraph on \( n \) vertices. If \( \delta(M) \geq 2(1/2)n - 1 \), then there exists a disjoint collection of fit \( s \)-cliques that tile all but at most \( s(s - 1)(2s - 1)/3 \) vertices of \( M \).

**Proof.** Let \( \mathcal{M} \) be a set of disjoint fit cliques in \( M \), each having at most \( s \) vertices. Let \( p_i \) be the number of \( i \)-cliques in \( \mathcal{M} \) and pick \( \mathcal{M} \) so that \( (p_s, \ldots, p_1) \) is maximized lexicographically.

Put \( \mathcal{Y} := \{Y \in \mathcal{M} : |Y| = s\} \) and \( \mathcal{X} = \mathcal{M} - \mathcal{Y} \). Set \( U := \bigcup_{X \in \mathcal{X}} V(X) \), \( W := \bigcup_{Y \in \mathcal{Y}} V(Y) \).

Assume, for a contradiction, that \( |U| > s(s - 1)(2s - 1)/3 \). We claim that for all \( X, X' \in \mathcal{X} \) with \( |X| \leq |X'| \), \( \|X, X'\| \leq 2\frac{s - s - 2}{s - 1}\|X'||X| \). If \( X = X' \), then \( \|X, X'\| \leq 2(|X| - 1)|X| \leq 2\frac{s - s - 2}{s - 1}\|X^2\| \).

If \( X \neq X' \), then the maximality of \( \mathcal{M} \) implies \( x + X' \) is not a fit \(|X'| + 1\)-clique for any \( x \in X \). Thus

\[
\|X, X'\| \leq \begin{cases} 
(2|X'| - 1)|X| = 2\frac{2|X'|-1}{2|X'|}|X'||X| \leq 2\frac{s - s - 2}{s - 1}|X'||X| & \text{if } |X'| \leq \frac{s - 1}{2}; \\
(2|X'| - 2)|X| = 2\frac{2|X'|-1}{|X'|}|X'||X| \leq 2\frac{s - s - 2}{s - 1}|X'||X| & \text{if } \frac{s}{2} \leq |X'| \leq s - 1.
\end{cases}
\]

Therefore by the claim,

\[
\|X, U\| \leq 2\frac{s - s - 2}{s - 1}|U||X| = 2\frac{s - s - 1}{s}|U||X| - \frac{2}{s(s - 1)}|U||X| < 2\frac{s - s - 1}{s}|U||X| - |X|.
\]

By the degree condition,

\[
\|X, W\| > 2\frac{s - s - 1}{s}|W||X|.
\]

Since

\[
\sum_{t=1}^{s-1} 2t^2 = \frac{s(s - 1)(2s - 1)}{3} < |U| = \sum_{t=1}^{s-1} tp_t,
\]

there exists \( t \in [s - 1] \) with \( p_t \geq 2t + 1 \). Choose \( \mathcal{X}' \subseteq \mathcal{X} \) such that \( |\mathcal{X}'| = 2t + 1 \) and \( |X| = t \) for every \( X \in \mathcal{X}' \). Put \( \mathcal{U}' := \bigcup_{X \in \mathcal{X}'} V(X) \).

By (4.2), there exists \( Y \in \mathcal{Y} \) such that \( \|U', Y\| \geq 2\frac{s - s - 1}{s}|U'||Y| + 1 = 2(s - 1)t|\mathcal{X}'| + 1 \). Let \( X_1, \ldots, X_{2t+1} \) be an ordering of \( \mathcal{X}' \) such that \( \|X_i, Y\| \geq \|X_{i+1}, Y\| \) for \( i \in [2t] \). Clearly, \( 2(s - 1)t + 2t \geq \|X_1, Y\| \geq (2s - 1)t + 1 \), so

\[
\|U' - V(X_1) , Y\| \geq (2s - 1)t(2t + 1) + 1 - (2(s - 1)t + 2t) = (2s - 1)t - 1)2t + 1.
\]

This implies \( \|X_2, Y\| \geq 2(s - 1)t \). Lemma 22 applied to \( X_1, X_2 \) and \( Y \) then gives a contradiction to the maximality of \( \mathcal{M} \). \[\square\]

The next lemma, adapted from an argument in [19], is probabilistic. It requires the union bound, the linearity of expectation, Markov’s inequality and Chernoff’s inequality [11]. For a \( d \)-tuple \( T := (v_1, \ldots, v_d) \in V^d \), let \( \text{im}(T) := \{v_1, \ldots, v_d\} \) denote the image of \( T \).
Lemma 24. Let \( m, d \in \mathbb{N}, \varphi > 0, \beta \in (0, \frac{\varphi}{2\beta}) \) and \( \gamma \in (0, 2\beta \left( \frac{\varphi}{2\beta} - \beta \right)) \). There exists \( n_0 \) such that when \( V \) is a set of order \( n \geq n_0 \) the following holds. For every \( S \in (V)_m \), let \( f(S) \) be a subset of \( V^d \). Call \( T \in V^d \) an absorbing tuple if \( T \in f(S) \) for some \( S \in (V)_m \). If \( |f(S)| \geq \varphi n^d \) for every \( S \in (V)_m \), then there exists a set \( F \) of at most \( \beta n/d \) absorbing tuples such that \( |f(S) \cap F| \geq \gamma n \) for every \( S \in (V)_m \) and the images of distinct elements of \( F \) are disjoint.

Proof. Pick \( \varepsilon > 0 \) so that

\[
(1 + \alpha)\varepsilon < \frac{\alpha \beta}{d} - 2\beta^2 - \gamma.
\]

Let \( \beta' := \frac{\beta}{d}, p := \beta' - \varepsilon \) and \( \gamma' := \gamma + (d^2 + 1)p^2 \). Let \( F' \) be a random subset of \( V^d \) where each \( T \in V^d \) is selected independently with probability \( pn^{1-d} \). Let

\[
\mathcal{O} := \left\{ \{T, T'\} \in \binom{V^d}{2} : \text{im}(T) \cap \text{im}(T') \neq \emptyset \right\}
\]

and \( \mathcal{O}_{F'} := \mathcal{O} \cap (\binom{F'}{2}) \).

We only need to show that, for sufficiently large \( n_0 \), with positive probability \( |\mathcal{O}_{F'}| < (d^2 + 1)p^2 n, |\mathcal{F}'| < \beta' n \) and \( |f(S) \cap \mathcal{F}'| > \gamma' n \) for every \( S \in (V)_m \). Indeed, we can then remove at most \( (d^2 + 1)p^2 n \) tuples from such a set \( F' \) so that the images of the remaining tuples are disjoint. The resulting set will satisfy (a), (c) and (d). To also satisfy (b), also remove every \( T \in F' \) for which there does not exist \( S \in (V)_m \) such that \( T \in f(S) \).

Clearly,

\[
|\mathcal{O}| \leq n \cdot d^2 \cdot n^{2d-2} = d^2 n^{2d-1},
\]

so for any \( \{T, T'\} \in \binom{V^d}{2} \), \( \Pr(\{T, T'\} \in \mathcal{F}') = p^2 n^{2-2d} \). Therefore, by the linearity of expectation, \( \mathbb{E}[|\mathcal{O}_{F'}|] < d^2 p^2 n \). So, by Markov's inequality,

\[
\Pr(|\mathcal{O}_{F'}| \geq (d^2 + 1)p^2 n) \leq \frac{d^2}{d^2 + 1}.
\]

Note that \( \mathbb{E}[|\mathcal{F}'|] = pn \) and \( pn \geq \mathbb{E}[|f(S) \cap \mathcal{F}'|] \geq \alpha p n \) for every \( S \in (V)_m \). Therefore, by Chernoff's inequality, \( \Pr(|\mathcal{F}'| \geq \beta' n) \leq \exp(-\varepsilon^2 n/3) \), and, since

\[
\alpha p - \gamma' = \frac{\alpha \beta}{d} - \alpha \varepsilon - (d^2 + 1) \left( \frac{\beta}{d} - \varepsilon \right)^2 - \gamma > \frac{\alpha \beta}{d} - 2\beta^2 - \gamma - \alpha \varepsilon > \varepsilon,
\]

\( \Pr(|\mathcal{F}' \cap f(S)| \leq \gamma' n) \leq \exp(-\varepsilon^2 n/3) \) for every \( S \in (V)_m \). Therefore, for sufficiently large \( n_0 \),

\[
\Pr(|\mathcal{O}_{F'}| \geq (d^2 + 1)p^2 n + |\mathcal{F}'| \geq \beta' n) + \sum_{S \in (V)_m} \Pr(|\mathcal{F}' \cap f(S)| \leq \gamma' n) < 1. \]

Lemma 25. Let \( s \geq 2, \varepsilon > 0, \) and \( M = (V, E) \) be a standard multigraph on \( n \) vertices. If \( \delta(M) \geq 2^{\frac{s-1}{s}} n + \varepsilon n \), then for all distinct \( x_1, x_2 \in V \), there exists \( A \subseteq V^{s-1} \) such that \( |A| \geq (\varepsilon n)^{s^{-1}} \) and for every \( T \in A \) both \( \text{im}(T) + x_1 \) and \( \text{im}(T) + x_2 \) are near matching \( s \)-cliques.
Proof. For $0 \leq t \leq s - 1$, the $t$-tuple $T \in V^t$ is called useful if, for both $i \in \{1, 2\}$, $\text{im}(T) + x_i$ is a near matching and $\|x_i, \text{im}(T)\|^t \leq \max\{0, t - s + 3\}$. To complete the proof we will show that there exists a set $A \subseteq V^{s-1}$ such that $|A| \geq (\varepsilon n)^{s-1}$ and every $T \in A$ is useful. Suppose there is no such set.

Let $0 \leq t < s - 1$ be the maximum integer for which there exists $A \subseteq V^t$ such that $|A| \geq (\varepsilon n)^t$ and every $T \in A$ is useful (note that 0 is a candidate since the empty function $f : \emptyset \rightarrow V$ is useful and $V^0 = \{f\}$); select $A$ so that in addition $\sum_{T \in A} \|M[\text{im}(T)]\|$ is maximized. Since $t$ is maximized, there exists $(v_1, \ldots, v_t) \in A$ with less than $\varepsilon n$ extensions $(v_1, \ldots, v_t, v)$ to a useful $(t + 1)$-tuple.

Let $m := n/s, Y := \{x_1, x_2, v_1, \ldots, v_t\}$, and $V_c := \{v \in V : \|v, Y\| \geq 2t + 4 - c\}$. Then $|V_0| \leq \varepsilon n$, since each $v \in V_0$ extends $(v_1, \ldots, v_t)$. Define

$$Z := \begin{cases} V_1 \cap N_H(x_1) \cap N_H(x_2) & \text{if } t \leq s - 4 \\ V_1 & \text{if } s - 3 \leq t \leq s - 2 \end{cases}$$

We claim that $|Z| \geq (t + 1)\varepsilon n$. Since $(t + 2)(2s - 2)m + (t + 2)\varepsilon n \leq \|Y, V\| \leq |V_0| + |V_1| + (2t + 4 - 2)sn, we have

$$|V_1| \geq 2(s - 2 - t)m + (t + 2)\varepsilon n - |V_0| \geq (t + 1)\varepsilon n. \quad (4.3)$$

So we are done unless $t \leq s - 4$. In this case, note that $|N_H(x_i)| \geq (s - 2)m + \varepsilon n$ for $i \in \{1, 2\}$, which combined with (4.3) gives

$$|Z| \geq 2(s - 2 - t)m + (t + 1)\varepsilon n - 4m \geq (t + 1)\varepsilon n.$$

So there exists $z \in Z \subseteq V_1$ such that $(v_1, \ldots, v_t, z)$ is not useful. Let $\{y\} = N_L(z) \cap Y$. The definitions of useful and $Z$ imply $y \notin \{x_1, x_2\}$ and $\|y, Y\| = 1$. But then $\|Y - y + z\| > \|Y\|$, contradicting the maximality of $\sum_{T \in A} \|M[\text{im}(T)]\|$. \hfill $\square$

Proof of Theorem 20. Assume $s \geq 2$ as otherwise the theorem is trivial. Let $d := s^2$ and $\alpha := \frac{s^d}{2}$. For any $S \in \binom{V}{s}$ call $Z \in \binom{V - S}{d}$ an $S$-sponge if both $M[Z]$ and $M[Z \cup S]$ have a perfect acceptable $s$-clique tiling. Define $f : \binom{V}{s} \rightarrow 2^{V^d}$ by

$$f(S) := \{T \in V^d : \text{im}(T) \text{ is an } S\text{-sponge}\}.$$

![Figure 4.1](image-url)

**Figure 4.1.** An $S$-sponge. Note that the tuples indicated by the dashed lines form a tiling and the tuples indicated by the solid lines form a larger tiling.
Claim. \(|f(S)| \geq \alpha n^d\) for every \(S \in \binom{V}{s}\).

Proof. Let \(S := \{x^i_1, \ldots, x^i_s\} \in \binom{V}{s}\). By Lemma 25 there are (many) more than \((\varepsilon n)^s\) tuples \(T_0 \in V^s\) such that \(im(T_0)\) is an acceptable \(s\)-clique and \(im(T_0) \cap S = \emptyset\). Let \((z^i_1, \ldots, z^i_s)\) be one such tuple. Again by Lemma 25, for every \(i \in [s]\) there are at least \((\varepsilon n)^{s-1}\) tuples \(T_i = (z^i_2, \ldots, z^i_s) \in V^{s-1}\) such that \(x^i_1 + im(T_i)\) and \(z^i_s + im(T_i)\) are both acceptable \(s\)-cliques. Therefore, when \(n\) is sufficiently large there are at least \((\varepsilon n)^s((\varepsilon n)^{s-1}) \geq \alpha n^d\) tuples \(T := (z^1_1, \ldots, z^s_1, \ldots, z^1_s, \ldots, z^s_s)\) such that if we define \(Z := im(T)\), \(Z_0 := \{z^1_1, \ldots, z^s_1\}\) and \(Z_i := \{z^i_2, \ldots, z^i_s\}\) for every \(i \in [s]\), then

- \(Z \in \binom{V - S}{d}\);
- \(\{z^i_1 + Z_i : i \in [s]\}\) is a perfect acceptable \(s\)-clique tiling of \(M[Z]\); and
- \(Z_0 + \{x^i_1 + Z_i : i \in [s]\}\) is a perfect acceptable \(s\)-clique tiling of \(M[Z \cup S]\). \(\square\)

Let \(\gamma, \beta < \min\{\alpha, \frac{1}{3}\}\) be constants that satisfy the hypothesis of Lemma 24, and let \(F \subset V^d\) be a set guaranteed by the lemma. Let \(Q := \bigcup_{T \in F} im(T)\) and note that \(|Q| = d|F| < \frac{1}{3}n\). Let \(M' := M - Q\).

We now can apply Theorem 23 to \(M'\) to tile all of the vertices of \(M'\) with acceptable \(s\)-cliques except a set \(X\) of order at most \(s(s - 1)(2s - 1)/3\). Partition \(X\) into sets of size \(s\). If \(n\) is sufficiently large, \(|X| \leq s\gamma n\). Therefore, for every set \(S\) in the partition of \(X\), we can choose a unique \(T \in f(S) \cap F\). This implies that there is a perfect acceptable \(s\)-clique tiling of \(M[X \cup Q]\) which completes the proof. \(\square\)

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