On Pebbling Threshold Functions for Graph Sequences
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Abstract

Given a connected graph $G$, and a distribution of $t$ pebbles to the vertices of $G$, a pebbling step consists of removing 2 pebbles from a vertex $v$ and placing one pebble on a neighbor of $v$. For a particular vertex $r$, the distribution is $r$-solvable if it is possible to place a pebble on $r$ after a finite number of pebbling steps. The distribution is solvable if it is $r$-solvable for every $r$. The pebbling number of $G$ is the least number $t$ so that every distribution of $t$ pebbles is solvable. In this paper we are not concerned with such an absolute guarantee but rather an almost sure guarantee. A threshold function for a sequence of graphs $\mathcal{G} = \{G_1, G_2, \ldots, G_n, \ldots\}$, where $G_n$ has $n$ vertices, is any function $t_0(n)$ such that almost all distributions of $t$ pebbles are solvable when $t \gg t_0$, and such that almost none are solvable when $t \ll t_0$.

We give bounds on pebbling threshold functions for the sequences of cliques, stars, wheels, cubes, cycles and paths.

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1 Introduction

All graphs considered here are connected and vertex-labeled. Given a graph $G_n$ on $n$ vertices and a distribution $D : V(G_n) \rightarrow \mathbb{N}$ of $t = \sum_v D(v)$ pebbles to the vertices of $G_n$, a pebbling step consists of removing 2 pebbles from a vertex $v$ and placing one pebble on a neighbor of $v$. For a particular root vertex $r$, the distribution is $r$-solvable if it is possible to place a pebble on $r$ after a finite number of pebbling steps. The distribution is $G_n$-solvable or just solvable if it is $r$-solvable for every $r$. The pebbling number $f(G_n)$ is the least integer $t = t(n)$ such that every distribution of $t$ pebbles is solvable. We always take the vertex set of an $n$-vertex graph to be $\{v_i | i \in [n]\}$ or $[n]$, where $[n] = \{1, \ldots, n\}$. Thus, distributions $D$ are independent of $G_n$.

Graph pebbling has an interesting history (see [3, 5]). Essentially it was invented by Lagarias and Saks to provide an elegant solution to a number-theoretic question of Erdős and Lemke, originally solved by Kleitman and Lemke [10]. The area has grown considerably, with many nice results, open problems, and difficult conjectures (see [7, 11]), including the famous conjecture of Graham that the pebbling number of a (cartesian) product of graphs is at most the product of the pebbling numbers of the graphs.

In this paper, we introduce a probabilistic pebbling model, where the pebbling distribution is selected uniformly at random from the set of all distributions with a prescribed number $t$ of pebbles. Since the distribution
of pebbles to vertices is like the distribution of unlabeled balls to labeled urns, our sample space consists of \( \binom{n+t-1}{t} \) (equally likely) outcomes. We define and study thresholds so that if \( t \) is essentially larger than the threshold, then any distribution is almost surely solvable, and if \( t \) is essentially smaller than the threshold, then any distribution is almost surely unsolvable. Of course, the definition mimics the important threshold concept in random graph theory. Unlike the situation in random graphs, however, it does not seem obvious that even “natural” families of graphs have pebbling thresholds. One candidate for such a family is the sequence of paths. We should emphasize also that, unlike in random graph theory, even the most basic random variables considered here are functions of dependent random variables, and the dependence is not “sparse”. This substantially limits the set of tools available for analyzing these random variables.

We now recall some basic asymptotic notation. For two functions \( f \) and \( g \), we write \( f \ll g \) (equivalently \( g \gg f \)) when the ratio \( f(n)/g(n) \) approaches 0 as \( n \) tends to infinity. We use \( o(g) \) and \( \omega(f) \), respectively, to denote the sets \( \{f \mid f \ll g\} \) and \( \{g \mid f \ll g\} \), so that \( f \in o(g) \) if and only if \( g \in \omega(f) \). In addition, we write \( f \in O(g) \) (equivalently \( g \in \Omega(f) \)) when there are positive constants \( c \) and \( k \) such that \( f(n)/g(n) < c \) for all \( n > k \), and we write \( \Theta(g) \) for \( O(g) \cap \Omega(g) \). We also use the shorthand notation \( \Theta(f) \leq \Theta(g) \) to mean that \( f' \in O(g') \) for every \( f' \in \Theta(f) \) and \( g' \in \Theta(g) \). Finally, we write \( f \lesssim g \) for \( \limsup f/g \leq 1\). To avoid cluttering the paper with floor and ceiling
symbols, we adopt the convention that large constants (such as $1/\epsilon$ when $\epsilon$ is small) are integers.

We are almost ready to define formally our notion of a pebbling threshold function. Let $D_n : [n] \to \mathbb{N}$ denote a distribution of pebbles on $n$ vertices. For a particular function $t = t(n)$, we consider the probability space $\Omega_{n,t}$ of all distributions $D_n$ of size $t$, i.e. with $t = \sum_{i \in [n]} D_n(i)$ pebbles. Given a graph sequence $\mathcal{G} = (G_1, \ldots, G_n, \ldots)$, denote by $P_{\mathcal{G}}(n, t)$ the probability that an element of $\Omega_{n,t}$ chosen uniformly at random is $G_n$-solvable. We call a function $g$ a threshold for $\mathcal{G}$, and write $g \in \text{th}(\mathcal{G})$, if the following two statements hold as $n \to \infty$: (i) $P_{\mathcal{G}}(n, t) \to 1$ whenever $t \gg g$, and (ii) $P_{\mathcal{G}}(n, t) \to 0$ whenever $t \ll g$. Notice that $\text{th}(\mathcal{G}) = \Theta(g)$ exactly when $g \in \text{th}(\mathcal{G})$.

We shall consider the following families of graphs.

- $\mathcal{K} = (K_1, \ldots, K_n, \ldots)$: $K_n$ is the complete graph on $n$ vertices.
- $\mathcal{P} = (P_1, \ldots, P_n, \ldots)$: $P_n$ is the path on $n$ vertices.
- $\mathcal{C} = (C_1, \ldots, C_n, \ldots)$: $C_n$ is the cycle on $n$ vertices.
- $\mathcal{S} = (S_1, \ldots, S_n, \ldots)$: $S_n$ is the star on $n$ vertices.
- $\mathcal{W} = (W_1, \ldots, W_n, \ldots)$: $W_n$ is the wheel on $n$ vertices.
- $\mathcal{Q} = (Q^1, \ldots, Q^m, \ldots)$: $Q^m$ is the $m$-dimensional cube on $n = 2^m$ vertices.
In addition, we will denote generic families of graphs by $\mathcal{G} = (G_1, \ldots, G_n, \ldots)$ or $\mathcal{H} = (H_1, \ldots, H_n, \ldots)$. We use $x \sim y$ to indicate that the vertices $x$ and $y$ are adjacent, so, e.g., in $P_n$, we have $v_i \sim v_j$ if and only if $j = i \pm 1$, and in $C_n$ we add the extra adjacency $v_1 \sim v_n$.

Supposing that thresholds exist for the graph sequences above, we will show that for every $\epsilon > 0$, we have $th(\mathcal{G}) \subseteq o(n^{1+\epsilon})$. For paths, we will show that $th(\mathcal{P}) = \Omega(n)$ which shows that $th(\mathcal{P}) \subseteq \Omega(n) \cap o(n^{1+\epsilon})$ for any $\epsilon > 0$. The technique used for paths can be extended to cycles, yielding a similar result. We further prove that $th(\mathcal{S}) = \Theta(n^{1/2})$, which implies a similar result for wheels. It is interesting to note that, in light of the results for paths and stars, it is conceivable that the set of thresholds for all possible sequences of trees may span the entire range of functions from $n^{1/2}$ to $n$ (or $n^{1+\epsilon}$, as the case may be).

The rest of the paper is organized as follows. In the next section we make a few observations concerning pebbling thresholds of general graphs. Section 3 contains an overview of some relevant pebbling theorems and their consequences for our probabilistic model. In Section 4, we establish the upper bound in the preceding paragraph as well as the results for paths, cycles, stars and wheels. The result for paths, Theorem 4.2, invokes Markov’s inequality, so we did not require the second moment. Nevertheless, we did determine the relevant variance with the hope of improving Theorem 4.2. This computation appears as an appendix in our final Section 6. In Section 5, we close the main
body of the paper with some intriguing open questions and problems.

2 Preliminaries

Here we make some elementary observations about pebbling thresholds of
generic families of graphs. Our first fact follows easily from the definitions.

**Fact 2.1** If $E(H_n) \subseteq E(G_n)$, then the probability that $D_n$ solves $G_n$ is at
least as large as the probability that $D_n$ solves $H_n$.

*Proof.* Every $H_n$-solvable distribution is $G_n$-solvable. \hfill \square

A simple consequence of this is

**Fact 2.2** If $E(H_n) \subseteq E(G_n)$ for all $n$, and $\text{th}(G)$ and $\text{th}(H)$ both exist, then
$\text{th}(G) \leq \text{th}(H)$.

A natural generalization leads to the following innocent looking

**Question 2.3** If $f(G_n) \leq f(H_n)$ for all $n$, and $\text{th}(G)$ and $\text{th}(H)$ both exist,
then is it true that $\text{th}(G) \leq \text{th}(H)$?

One approach to answering Question 2.3 affirmatively might be to at-
ttempt to prove that, if $f(G_n) \leq f(H_n)$, then, for all $t$, $P_{G}(n,t) \geq P_{H}(n,t)$.
However, this statement is false, as the following example shows. In Figure 1,
Figure 1: A counterexample to \( f(G_n) \leq f(H_n) \Rightarrow P_v(n, t) \geq P_H(n, t) \).

besides participating in a \( K_4 \) with the vertices \( b, c, d \in V(G_n) \), the vertex \( a \) is joined to every vertex in the two copies of \( K_k \) indicated by line segments, and likewise for \( b, c \) and \( d \).

It was proved in [12] that every diameter 2 graph has pebbling number either \( n \) or \( n + 1 \). In [5], the family \( \mathcal{F} \) of diameter 2 graphs having pebbling number \( n + 1 \) was characterized. Since the diameter of \( G_n \) is 2 and \( G_n \notin \mathcal{F} \), \( f(G_n) = n \). It is easy to see that every graph with a cut vertex has pebbling number at least \( n + 1 \), and since \( H_n \) is such a graph and has diameter 2, \( f(H_n) = n + 1 \). In \( G_n \), if \( D = D_n \) has \( D(r) = D(a) = D(b) = D(c) = D(d) = 0 \), \( D(u) = D(v) = 3 \), and \( D(x) = 1 \) otherwise, then \( D \) is \( r \)-unsolvable in \( G_n \) and has \( t = n - 1 \) pebbles. The number of such distributions is in \( \Omega(n^3) \).

In \( H_n \), if \( D = D_n \) has \( D(r) = D(a) = D(b) = 0 \), \( D(u) = 3 \), and \( D(x) = 1 \) otherwise, then \( D \) is \( r \)-unsolvable in \( H_n \) and has \( t = n - 1 \) pebbles. The
number of such distributions is in $\Theta(n^2)$, and these are the only distributions which do not solve $H_n$. Hence $P_G(n, t) \leq P_H(n, t)$.

However, one can imagine that the implication in Question 2.3 may hold with the added hypothesis that $f(G_n)$ is significantly smaller than $f(H_n)$: say $f(H_n) - f(G_n) \to \infty$ as $n \to \infty$, or $\limsup_{n \to \infty} f(G_n)/f(H_n) < 1$.

## 3 Precursors

Our first threshold result is for cliques. Because of the Pigeonhole Principle it is merely an “unordered” reformulation of the so-called “Birthday Problem” (see [13]).

### Result 3.1 (Clarke, Hurlbert)[4] The threshold for the sequence of cliques is $\text{th}(K) = \Theta(n^{1/2})$.

**Proof.** If $D = D_n$ is chosen uniformly at random from among all $\binom{n+t-1}{t}$ distributions on $K_n$ with $t = t(n)$ pebbles, then the favorable distributions for the event $B = \{D$ is not solvable$\}$ are those in which the pebbles occupy distinct vertices. Thus,

$$\Pr(B) = \frac{\binom{n}{t}}{\binom{n+t-1}{t}} = \frac{n!}{(n+t-1)!} \frac{(n+t-1)!}{(n+t)!} = \frac{1}{(n+t)}.$$

which, by Stirling’s formula, approaches 1 if $t(n) \ll n^{1/2}$ and 0 if $t(n) \gg n^{1/2}$.

Since $P_k(n, t) = 1 - \Pr(B)$, the result follows. $\square$
Next, we note the following obvious fact.

**Fact 3.2** For given $G$ define the function $g : \mathbb{N} \rightarrow \mathbb{N}$ by $g(n) = f(G_n)$ and suppose that $th(G)$ exists. Then $th(G) \subseteq O(g)$. □

This yields the following easy corollaries.

**Corollary 3.3** If $th(Q)$ exists then $th(Q) \subseteq O(n)$.

*Proof.* Chung proved in [3] that $f(Q^m) = n = 2^m$. □

**Corollary 3.4** Let $G_n$ have diameter 2 for all $n$. If $th(G)$ exists then $th(G) \subseteq O(n)$.

*Proof.* It was proved in [12] that $f(G_n) \leq n + 1$ (see also [5]). □

In fact, Corollary 3.4 can be generalized by noting that, for all $G_n$, $f(G_n) \leq 2^d n$, where $d$ is the diameter of $G_n$ (this follows easily from the Pigeonhole Principle). Then Fact 3.2 also yields

**Corollary 3.5** Let $d(n) = \text{diameter}(G_n)$ and suppose that $th(G)$ exists. Then $th(G) \subseteq O(2^{d(n)} n)$. In particular, if $d(n) \leq d$ for all $n$, then $th(G) \subseteq O(n)$.

More recently, it was proved in [6] that $f(G) = n$ for any graph $G$ for which $k \geq 2^{2d+3}$, where the connectivity of $G$ is at least $k$ and the diameter of $G$ is at most $d$. 
Corollary 3.6 Let $d(n) = \text{diameter}(G_n)$, $k(n) = \text{connectivity}(G_n)$, and suppose that $th(G)$ exists. If $k(n) \geq 2^{2d(n)+3}$ for all $n$, then $th(G) \subseteq O(n)$.

4 Thresholds

We begin with an upper bound on $th(G)$, valid for all $G$. Then we present our theorems on pebbling thresholds for paths and stars, followed by applications to the thresholds for cycles and wheels.

Theorem 4.1 For all $G$, if $th(G)$ exists then, for any given $\epsilon > 0$, we have $th(G) \subseteq o(n^{1+\epsilon})$.

Proof. Let $\epsilon > 0$ be given, and let $t : \mathbb{N} \rightarrow \mathbb{N}$ so that, for each $n$, $D = D_n$ is chosen uniformly at random from among all distributions on $[n]$ of size $t(n)$. Denote by $\Pr(n)$ the probability that $D$ is solvable. We show that $t(n) \in \Omega(n^{1+\epsilon})$ implies $\Pr(n) \rightarrow 1$ as $n \rightarrow \infty$.

One can argue that, for any graph $H_l$ on $l$ vertices, we have $f(H_l) < 2^l$. Indeed, suppose $T_l$ is a spanning tree of $H_l$, and let $P_l$ be the path on $l$ vertices. Then $f(H_l) \leq f(T_l) \leq f(P_l) = 2^{l-1}$. The last inequality follows from a result of Moews [11]. We use this observation below.

Let $\delta > 0$, $t = cn^{1+\epsilon}$ for some $c > 0$, $l = (1 + \delta)/\epsilon$, and $k = 2^l$. Fix $n$ and consider the graph $G = G_n$. For each vertex $v$, choose $G(v)$ to be a connected subgraph of $G$ containing $l$ vertices, including $v$. Let $|D_{G(v)}|$ denote
the number of pebbles on vertices of $G(v)$. We call $G(v)$ an \textit{l-neighborhood}

of $G$; the subgraph $G(v)$ is \textit{k-bounded} in case $|D_{G(v)}| < k$. We claim that the

probability that there is a $k$-bounded $l$-neighborhood tends to zero. Indeed,

\[
\Pr \left[ \exists \text{ } k\text{-bounded } l\text{-neighborhood} \right] \leq n \Pr \left[ G(v) \text{ is } k\text{-bounded} \right]
\]

\[
= n \sum_{i=0}^{k-1} \Pr \left[ |D_{G(v)}| = i \right] = n \sum_{i=0}^{k-1} \frac{(l+i-1)(n-l+t-i-1)}{(n+t-1)} \frac{t}{n-l+t-i-1}
\]

\[
= \frac{n}{(n+t-1)} \sum_{i=0}^{k-1} \binom{l+i-1}{i} \binom{n-l+t-1}{t} \frac{t}{n-l+t-i-1} \prod_{j=0}^{i-1} \left( \frac{t-j}{n-l+t-j-1} \right)
\]

\[
\leq \frac{n}{(n+t-1)} \sum_{i=0}^{k-1} \binom{l+i-1}{i} \binom{n-l+t-1}{t} \left( \frac{t}{n-l+t-1} \right)^i \prod_{j=1}^{i} \left( \frac{n-j}{n+t-j} \right)
\]

\[
= \frac{n}{(n+t-1)} \sum_{i=0}^{k-1} \binom{l+i-1}{i} \left( \frac{t}{n-l+t-1} \right)^i \left( \frac{n}{t} \right)^i
\]

\[
= n \left( \frac{t}{n} \right)^i \sum_{i=0}^{k-1} \binom{l+i-1}{i} \left( \frac{t}{n-l+t-1} \right)^i
\]

\[
\leq n(n^{-d}) \sum_{i=0}^{k-1} \binom{l+i-1}{i}
\]

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\[
\leq n^{1-\epsilon} k \binom{l+k-1}{k}
\]

\[= C n^{-\delta} \to 0.\]

Thus, with probability tending to 1, every \(l\)-neighborhood of \(G\) contains at least \(k = 2^l\) pebbles, from which \(\Pr(n) \to 1\) follows. \(\square\)

**Theorem 4.2** If \(th(P)\) exists then, for every \(\epsilon > 0\), we have \(th(P) \subseteq \Omega(n) \cap o(n^{1+\epsilon})\).

**Proof.** The upper bound follows from Theorem 4.1, so it remains only to establish the lower bound. Let \(t : \mathbb{N} \to \mathbb{N}\) and, as usual, for each \(n\) let \(D = D_n\) be chosen uniformly at random from among all distributions of size \(t(n)\). Denote by \(\Pr(n)\) the probability that \(D\) is solvable. We show that \(t(n) \ll n\) implies \(\Pr(n) \to 0\) as \(n \to \infty\).

Let the vertices of \(P_n\) be labeled so that \(v_1 \sim v_2 \sim \cdots \sim v_n\). For each \(v_i\), let \(X_i = D(v_i)\); also, let \(X = \sum X_i\) and \(Y = \sum X_i/2^{i-1}\). Notice that \(P_n\) is \(v_1\)-solvable if and only if \(Y \geq 1\). We proceed to show that the probability of this event tends to 0 when \(t \ll n\).

Let \(t = n/\omega\) for any \(\omega \to \infty\). Then \(\mathbf{E}(X_i) = t/n = 1/\omega \to 0\). Consequently, \(\mathbf{E}(Y) = \sum \mathbf{E}(X_i)/2^{i-1} = (\sum 1/2^{i-1})/\omega < 2/\omega \to 0\). Now, Markov’s inequality (see e.g. [13] or [14]) shows that

\[
\Pr[Y \geq 1] \leq \mathbf{E}(Y)/1 \to 0. \quad \square
\]
Theorem 4.3 If \( th(C) \) exists then, for every \( \epsilon > 0 \), we have \( th(C) \subseteq \Omega(n) \cap o(n^{1+\epsilon}) \).

Proof. Label the vertices of \( C_n \) as above, with \( v_n \sim v_1 \). Use the same proof technique as in Theorem 4.2, except replace \( Y \) with \( Y + Y' \), where \( Y' = \sum X_i/2^{n-i+1} \). Then \( E(Y') = E(Y) \), so

\[
\Pr[Y + Y' \geq 1] \leq E(Y + Y') = 2E(Y) \to 0. \quad \square
\]

Theorem 4.4 The threshold for the sequence of stars is \( th(S) = \Theta(n^{1/2}) \).

Proof. It follows from Fact 2.2 and Result 3.1 that \( th(S) \subseteq \Omega(n^{1/2}) \), so we need only show that \( th(S) \subseteq O(n^{1/2}) \).

Let \( t : N \to N \), and for each \( n \) let \( D = D_n \) be chosen uniformly at random from among all distributions on \([n]\) of size \( t(n) \). Denote by \( \Pr(n) \) the probability that \( D \) is solvable. By considering the probability \( q = 1 - \Pr(n) \) of the complementary event that \( D \) is unsolvable, we show that \( t(n) \gg n^{1/2} \) implies \( \Pr(n) \to 1 \) as \( n \to \infty \).

Let \( t = t(n) = \omega n^{1/2} \) for any \( \omega \to \infty \). Because \( f(S_n) = n \) for all \( n \), we may assume that \( t < n \). We produce an upper bound for \( q \) that tends to \( 0 \) by over-counting the number of unsolvable distributions of size \( t \) and dividing by the total number \( \binom{n+t-1}{t} \) of distributions.

If \( D \) is unsolvable, then there is at most one vertex \( v \) with \( D(v) > 1 \). (Indeed, if \( c \) is the center of the star and both of \( D(u) \), \( D(v) \) exceed 1, then
it is possible to pebble, if necessary, so that \(c\) now has at least two pebbles. Then one can pebble to any root.) In addition, we have \(D(v) < 4\) for all \(v\) because each \(S_n\) has diameter 2.

The number of such distributions is equal to the number having no \(v\) with \(D(v) > 1\), plus the number having exactly one \(v\) (different from the center) with \(1 < D(v) < 4\). (In the latter case, the center contains no pebbles.) This number is exactly \(\binom{n}{t} + (n-1)\binom{n-2}{t-2} + (n-1)\binom{n-2}{t-3}\). Hence

\[
q = \left[1 + \frac{t(t-1)}{n} + \frac{t(t-1)(t-2)}{n(n-t+1)}\right] \frac{(n)_t}{(n+t-1)_t}
\]

\[
< \left[1 + \frac{t^2}{n} + \frac{t^3}{n(n-t)}\right] \left(\frac{n}{n+t}\right)^{t-1}
\]

\[
\approx \left[1 + \omega^2 + \frac{\omega^3 n^{1/2}}{n - \omega n^{1/2}}\right] e^{-\omega(t-1)/(n+t)}
\]

\[
\approx \left[1 + \omega^2 + \frac{\omega^3 n^{1/2}}{n - \omega n^{1/2}}\right] e^{-\omega^2/2}. \quad (1)
\]

To see that the last expression tends to zero, consider two cases. If \(\omega < n^{1/2} - 1\) (so that \(n - \omega n^{1/2} > n^{1/2}\)), then the right side of (1) is at most \([1 + \omega^2 + \omega^3] e^{-\omega^2/2}\), which approaches 0 as \(\omega \to \infty\). On the other hand, if \(\omega \geq n^{1/2} - 1\), then, since \(n - \omega n^{1/2} = n - t \geq 1\), the right side of (1) is at most \([1 + \omega^2 + \omega^3 n^{1/2}] e^{-\omega^2/2} \leq [1 + \omega^2 + \omega^3(\omega + 1)] e^{-\omega^2/2}\), which also tends to 0 as \(\omega \to \infty\). \(\square\)
Remark While the preceding case analysis has the advantage of keeping
the proof elementary, our decision to employ this strategy hides a cleaner
— and we believe more elegant — approach depending on a natural lattice-
theoretic property. The normal property of the multiset lattice (see e.g. [8]
or [9]) states that if \( \mathcal{F} \) is a monotone increasing family of multisets (sub-
multisets) of a fixed set, then \( \Pr(\mathcal{F}_t) \), the fraction of all multisets of size \( t \)
belonging to \( \mathcal{F} \), is an increasing function of \( t \). Of course, the family of all
solvable distributions—where a distribution with \( t \) pebbles is viewed as a
\( t \)-multiset of the vertex set—is monotone increasing, since it is closed under
the operation of adding an element (a pebble) to any of its members.

In this context, the normal property guarantees that \( \Pr(n) \) in the proof
of Theorem 4.4 increases with \( t \). Thus, for this proof, it would have sufficed
to consider only small values of \( t \), relative to \( n \), and the reader may easily
verify that the assumption \( \omega \ll n^{1/2} \) reduces to one the number of our closing
cases.

Fact 2.2—and comparison with stars and cliques—yields the

Corollary 4.5 The threshold for the sequence of wheels is \( \text{th}(\mathcal{W}) = \Theta(n^{1/2}) \).

\[ \square \]
5 Questions

To close, we add some questions and problems for further research to the list started with Question 2.3. Most fundamental is the

**Question 5.1** Is it true that for any graph sequence \( \mathcal{G} = \{G_1, \ldots, G_n, \ldots\} \), its pebbling threshold \( \text{th}(\mathcal{G}) \) exists?

Based on our results in Sections 3 and 4, it may seem obvious how to construct a sequence which would yield a negative answer to Question 5.1. For example, consider the graph sequence \( \mathcal{G} = (K_1, C_2, \ldots, K_{2m-1}, C_{2m}, \ldots) \) and let \( t : \mathbb{N} \to \mathbb{N} \) be such that \( n^{1/2} \ll t(n) \ll n / \log \log n \). Then the probability \( \Pr(n) \) that a uniformly randomly chosen distribution \( D_n \) of size \( t(n) \) is solvable has no limit as \( n \to \infty \). Indeed, as \( m \to \infty \), \( \Pr(2m - 1) \to 1 \) by Result 3.1, while \( \Pr(2m) \to 0 \) by Theorem 4.3. Nevertheless, \( \text{th}(\mathcal{G}) \) still exists; it is simply \( \text{th}(\mathcal{K}) \) for odd \( n \) and \( \text{th}(\mathcal{C}) \) for even \( n \).

**Problem 5.2** Find \( \text{th}(\mathcal{P}) \).

Because \( \Theta(n) \leq \text{th}(\mathcal{P}) \leq \Theta(n^{1+\epsilon}) \), there is room for threshold functions like \( n \log n \). However, because the variance of \( Y \) (from the proof of Theorem 4.2) is large—see the Appendix—there is also room for no threshold to exist. That is, it may be the case that for some \( t \) satisfying \( n \ll t \ll n^{1+\epsilon} \), we have

\[
0 < \liminf_{n \to \infty} P_{\mathcal{P}}(n, t) \leq \limsup_{n \to \infty} P_{\mathcal{P}}(n, t) < 1.
\]
Problem 5.3  Find \( th(Q) \).

Given that the pebbling numbers of the cubes are known exactly (see the
proof of Corollary 3.3 or [3]), it seems surprising that our present knowledge
leaves open such a wide gap between the lower and upper bounds for \( th(Q) \in \Omega(n^{1/2}) \cap O(n) \).

Finally, we ask

Question 5.4  Is it true that, for all \( \Omega(n^{1/2}) \ni t_1 \ll t_2 \in O(n) \), there is a
graph sequence \( G \) such that \( th(G) \in \Omega(t_1) \cap O(t_2) \)?

In view of Theorems 4.2 and 4.4 (respectively on thresholds for paths and
stars, the “extreme cases” of trees), it is conceivable that this is true even
within the class of trees.

6  Appendix

Here we compute the second moment of the random variable \( Y \), defined in
the proof of Theorem 4.2. Though it is not used explicitly in this paper, a
refinement of our ideas eventually led via the second moment method—see
[1]—to an improvement of Theorem 4.2. This is discussed in a forthcoming
paper [2].

Let us start with a few combinatorial identities. The first one is standard.
Lemma 6.1 \( \sum_{i=0}^{t} \binom{l+a}{a} = \binom{l+a+1}{a+1}. \)

Lemma 6.2 \( \sum_{i=0}^{t} t \binom{l+a}{a} = t \binom{a+1}{a+2} \binom{l+a+1}{a+1}. \)

Proof.
\[
\frac{1}{a+1} \sum_{i=0}^{t} \binom{l+a}{a} = \sum_{i=1}^{t} \binom{l+a}{a+1} = \sum_{i=0}^{t-1} \binom{l+1+a}{a+1}
\]
which, by Lemma 6.1, is equal to \( \binom{t+a+1}{a+2} = \frac{t}{a+2} \binom{t+a+1}{a+1} \). \( \square \)

Lemma 6.3 \( \sum_{i=0}^{t} l^2 \binom{l+a}{a} = (a+1)t \left( \frac{t-1}{a+3} + \frac{1}{a+2} \right) \binom{t+a+1}{a+1}. \)

Proof.
\[
\sum_{i=0}^{t} l^2 \binom{l+a}{a} = (a+1) \left( \sum_{i=1}^{t} l \binom{l+a}{a+1} = (a+1) \sum_{l=0}^{t-1} (l+1) \binom{l+a+1}{a+1} \right)
\]
\[
= (a+1) \left[ \sum_{l=0}^{t-1} l \binom{l+a+1}{a+1} + \sum_{l=0}^{t-1} \binom{l+a+1}{a+1} \right]. \tag{2}
\]
Now we apply Lemmas 6.1 and 6.2 to conclude that the right-hand side of (2) is equal to
\[
(a+1) \left[ (t-1) \left( \frac{a+2}{a+3} + 1 \right) \binom{t+a+1}{a+2} \right]
\]
\[
= (a+1)t \left( \frac{t-1}{a+3} + \frac{1}{a+2} \right) \binom{t+a+1}{a+1}. \] \( \square \)

Let \( Y = \sum_{i=1}^{n} Y_i \), where the \( Y_i = 2^{1-i} X_i \) are the random variables defined in Section 4. If we consider distributions with \( t \) pebbles, then \( E(X_i) = t/n \), whence \( E(Y) = \sum E(Y_i) = tn^{-1} \sum 2^{1-i} \sim 2t/n. \)
Lemma 6.4 \( E(X_i^2) = [2t^2 + t(n-1)]/n(n+1). \)

Proof.
\[
E(X_i^2) = \sum_{k=0}^{t} k^2 \frac{(t+n-k-2)}{\binom{t+n-1}{t}} = \frac{1}{\binom{t+n-1}{n-1}} \sum_{l=0}^{t} (t-l)^2 \binom{l+n-2}{n-2}
\]

\[
= \frac{1}{\binom{t+n-1}{n-1}} \left[ t^2 \sum_{l=0}^{t} \binom{l+n-2}{n-2} - 2t \sum_{l=0}^{t} l \binom{l+n-2}{n-2} + \sum_{l=0}^{t} l^2 \binom{l+n-2}{n-2} \right]
\]

and, applying Lemmas 6.1, 6.2 and 6.3 with \( a = n-2 \), we see that

\[
E(X_i^2) = t^2 - 2t^2 \left( \frac{n-1}{n} \right) + (n-1)t \left( \frac{t-1}{n^2} + \frac{1}{n} \right) = \frac{2t^2 + t(n-1)}{n(n+1)}. \quad \square
\]

Lemma 6.5 For \( i \neq j \), we have \( E(X_iX_j) = (t^2 - t)/n(n+1) \).

Proof. Since
\[
E(X_iX_j) = \sum_{k=0}^{t} k \frac{(t+n-k-2)}{\binom{t+n-1}{t}} E(X_j|X_i = k),
\]
and \( E(X_j|X_i = k) \) is equal to \( (t-k)/(n-1) \), it follows that

\[
E(X_iX_j) = \frac{1}{(n-1)\binom{t+n-1}{n-1}} \sum_{k=0}^{t} k(t-k) \binom{t-k+n-2}{n-2}
\]

\[
= \frac{1}{(n-1)\binom{t+n-1}{n-1}} \sum_{l=0}^{t} (t-l)t \binom{l+n-2}{n-2}
\]

\[
= \frac{1}{(n-1)\binom{t+n-1}{n-1}} \left[ \sum_{l=0}^{t} l \binom{l+n-2}{n-2} - \sum_{l=0}^{t} l^2 \binom{l+n-2}{n-2} \right]
\]

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which, by Lemmas 6.2 and 6.3, is equal to

\[
\left( \frac{1}{n-1} \right) \left[ \frac{(n - 1)t^2}{n} - (n - 1)t \left( \frac{t - 1}{n + 1} + \frac{1}{n} \right) \right] = \frac{t^2 - t}{n(n + 1)}. \quad \square
\]

At last we are ready to compute the second moment of $Y$:

\[
E(Y^2) = E \left( \left( \sum Y_i \right)^2 \right) = E \left( \sum Y_i^2 \right) + E \left( \sum_{i \neq j} Y_i Y_j \right)
\]

\[
= \sum \frac{E(X_i^2)}{4i-1} + \sum \frac{E(X_iX_j)}{2i+j-2}.
\]

Note that if $t \geq n$ and $n$ is large, we have $E(X_iX_j) \sim t^2/n^2$ and $2t^2n^{-2} \leq E(X_i^2) \leq 3t^2n^{-2}$. Thus, for $t \geq n$ and some positive constants $c_1$, $c_2$, we have

\[
c_1 \left( \frac{t}{n} \right)^2 \leq E(Y^2) \leq c_2 \left( \frac{t}{n} \right)^2.
\]
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References


