2-factors of bipartite graphs with asymmetric minimum degrees

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Abstract

Let $G$ and $H$ be balanced $U, V$-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. Let $k$ be the number of components of $H$, $\delta_U := \min\{\deg_G(u) : u \in U\}$ and $\delta_V := \min\{\deg_G(v) : v \in V\}$. We prove that if $n$ is sufficiently large and $\delta_U + \delta_V \geq n + k$ then $G$ contains $H$. This answers a question of Amar in the case that $n$ is large. We also show that $G$ contains $H$ even when $\delta_U + \delta_V \geq n + 2$ as long as $n$ is sufficiently large in terms of $k$ and $\delta(G) \geq \frac{n}{200k} + 1$.

1 Introduction

This paper is motivated by several lines of research. Let $C_n^r$ ($P_n^r$) be the $r$-th power of a cycle (path) on $n$ vertices $C_n$ ($P_n$). In attempt to inspire a new proof of the Hajnal-Szemerédi theorem, Seymour made the following conjecture:

Conjecture 1.1 (Seymour [18]). If $G$ is a graph on $n$ vertices with $\delta(G) \geq \frac{r}{r+1}n$, then $C_n^r \subseteq G$.

Note that the case $r = 1$ is Dirac’s Theorem and the case $r = 2$ is Pósa’s Conjecture. Komlós, Sárközy and Szemerédi [13, 14] have used Szemerédi’s Regularity Lemma [19] and their own Blow-up Lemma [12] to prove Seymour’s conjecture for huge graphs, however even Pósa’s Conjecture remains open for small graphs.

Chau generalized the minimum degree condition in Seymour’s conjecture to an Ore-type degree condition.

Conjecture 1.2 (Chau [6]). Suppose $G$ is a graph on $n$ vertices such that $\deg(x) + \deg(y) \geq \frac{n}{r+1} - \frac{1}{r+1}$ for all non-adjacent pairs of vertices $x, y \in V(G)$.

(i) If $\delta(G) = \frac{n+1}{r+1}n + 2$ or $\delta(G) = \frac{n+1}{r+1}n + \frac{5}{3}$, then $P_n^r \subseteq G$.

(ii) If $\delta(G) > \frac{n+1}{r+1}n + 2$, then $C_n^r \subseteq G$.

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When \( r = 1 \), the condition \( \deg(x) + \deg(y) \geq \frac{2n}{r+1} n - \frac{r-1}{r+1} \) is Ore’s condition and thus \( C_n^r \subseteq G \) with no further restrictions on the minimum degree. Chau proved Conjecture 1.2 for huge graphs when \( r = 2 \).

The following fundamental graph packing conjecture was made independently by Bollobás-Eldridge [4] and Catlin [5]. We state it here in a complementary form.

**Conjecture 1.3** (Bollobás-Eldridge [4], Catlin [5]). If \( G \) and \( H \) are graphs on \( n \) vertices with \( \Delta(H) \leq r \) and \( \delta(G) \geq \frac{rn - 1}{r+1} \), then \( H \subseteq G \).

Call a graph on \( n \) vertices \( r \)-universal if it contains every graph \( H \) on \( n \) vertices with \( \Delta(H) \leq r \), then Conjecture 1.3 states that \( G \) is \( r \)-universal if \( \delta(G) \geq \frac{rn - 1}{r+1} \). The case \( r = 1 \) follows from the path version of Dirac’s Theorem: Since \( \delta(G) \geq \frac{n-1}{2} \), \( G \) contains the 1-universal graph \( P_n \). Aigner and Brandt [2] proved Conjecture 1.3 for the case \( r = 2 \). Fan and Kierstead [10] proved the path version of Pósa’s Conjecture: If \( \delta(G) \geq \frac{2n-1}{3} \) then \( G \) contains the square \( P_n^2 \) of \( P_n \). Since \( P_n^2 \) is 2-universal, we have a stronger version of the Aigner-Brandt Theorem: If \( \delta(G) \geq \frac{2n-1}{3} \) then \( G \) contains a 2-universal graph with maximum degree 4. Csaba, Shokoufandeh and Szemerédi [7] have proved Conjecture 1.3 for large graphs when \( r = 3 \).

Kostochka and Yu generalized the minimum degree condition in the Bollobás-Eldridge conjecture to an Ore-type degree condition.

**Conjecture 1.4** (Kostochka-Yu [15]). If \( G \) and \( H \) are graphs on \( n \) vertices with \( \Delta(H) \leq r \) and \( \deg(x) + \deg(y) \geq \frac{2(n-1)}{r+1} \) for all non-adjacent pairs of vertices \( x, y \in V(G) \), then \( H \subseteq G \).

The case \( r = 1 \) follows from the path version of Ore’s theorem: Since \( \deg(x) + \deg(y) \geq n - 1 \) for all non-adjacent pairs of vertices \( x, y \in V(G) \), \( G \) contains the 1-universal graph \( P_n \). Kostochka and Yu [16] proved Conjecture 1.4 for the case \( r = 2 \).

El-Zahar made the following conjecture.

**Conjecture 1.5** (El-Zahar [9]). If \( G \) is a graph on \( n \) vertices with \( \delta(G) \geq \sum_{i=1}^{k} \left\lceil \frac{1}{2} n_i \right\rceil \) where \( n_i \geq 3 \) and \( n = \sum_{i=1}^{k} n_i \), then \( G \) contains \( k \) disjoint cycles of lengths \( n_1, \ldots, n_k \).

El-Zahar proved that if \( G \) is a graph on \( n \) vertices with \( \delta(G) \geq \left\lceil \frac{1}{2} n_1 \right\rceil + \left\lceil \frac{1}{2} n_2 \right\rceil \) where \( n_1, n_2 \geq 3 \) and \( n = n_1 + n_2 \), then \( G \) contains two disjoint cycles of lengths \( n_1 \) and \( n_2 \). Abassi [1] used the Blow-up and Regularity Lemmas to prove El-Zahar’s Conjecture for huge \( n \).

Now we focus our attention on bipartite graphs. A \( U, V \)-bigraph is balanced if \( |U| = |V| \). We will call a balanced bipartite graph on \( 2n \) vertices \( bi\text{-universal} \) if it contains every balanced bipartite graph \( H \) with \( |H| = 2n \) and \( \Delta(H) = 2 \). Wang made the following conjecture.

**Conjecture 1.6** (Wang [20]). Every balanced bipartite graph \( G \) on \( 2n \) vertices with \( \delta(G) \geq n/2 + 1 \) is bi-universal.

An \( n \)-ladder, denoted by \( L_n \), is a balanced bipartite graph with vertex sets \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_n\} \) such that \( a_i \sim b_j \) if and only if \( |i - j| \leq 1 \). We refer to the edges \( a_i b_i \) as rungs and the edges \( a_1 b_1, a_n b_n \) as the first and last rung respectively. It is easily checked that an \( n \)-ladder is a bi-universal graph with maximum degree 3. In this sense, a ladder in a bipartite graph is analogous to a square path in a graph. The first and last author [8] used the Blow-up and Regularity Lemmas to prove Conjecture 1.6 for huge graphs by proving that such graphs contain a spanning ladder.
Finally we consider bipartite graphs with asymmetric minimum degrees. For a $U, V$-bigraph $G$, let $\delta_U := \delta_U(G)$ and $\delta_V := \delta_V(G)$ denote the minimum degrees of vertices in $U$ and $V$ respectively. The number of components of $G$ is denoted by $\text{comp}(G)$. Moon and Moser [17] proved that if $G$ is a balanced bipartite graph on $2n$ vertices with $\delta_U + \delta_V \geq n + 1$, then $G$ is hamiltonian. Amar [3] proved the following result about more general 2-factors. If $G$ and $H$ are balanced $U, V$-bigraphs on $2n$ vertices with $\delta_U + \delta_V \geq n + 2$, $\Delta(H) \leq 2$ and $\text{comp}(H) \leq 2$ then $G$ contains $H$. As noted in [3], when $\text{comp}(H) \leq 2$ this result is best possible. Amar then made the following conjecture.

Conjecture 1.7 (Amar [3]). Let $G$ and $H$ be balanced $U, V$-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. If $\delta_U + \delta_V \geq n + \text{comp}(H)$ then $G$ contains $H$.

We will prove the following theorems, strengthening Conjecture 1.7 for huge graphs.

Theorem 1.8. Let $G$ and $H$ be balanced $U, V$-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. For every integer $k$ there exists $N_0(k)$ such that if $n \geq N_0(k)$, $\delta_U + \delta_V \geq n + 2$, and $\text{comp}(H) \leq k$, then $G$ contains $H$. Furthermore, if $\delta(G) \geq \frac{1}{200k}n + 1$ then $G$ contains a spanning ladder.

Theorem 1.9. There exists a constant $C$ such that every balanced $U, V$-bigraph $G$ on $2n$ vertices satisfying $\delta_U + \delta_V \geq n + C$ contains a spanning ladder.

Theorem 1.10. Let $G$ and $H$ be balanced $U, V$-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. There exists an integer $N_0$ such that if $n \geq N_0$ and $\delta_U + \delta_V \geq n + \text{comp}(H)$ then $G$ contains $H$.

We note that there are no known counterexamples to show that the bound in Amar’s conjecture is tight when $k \geq 3$. In fact, Wang made the following stronger conjecture:

Conjecture 1.11 (Wang [21]). Every balanced $U, V$-bigraph on $2n$ vertices with $\delta_U + \delta_V \geq n + 2$ is bi-universal.

In Theorem 1.10 we prove Amar’s conjecture for huge graphs, but Theorem 1.8 gives evidence to suggest that a proof of Conjecture 1.11 should ultimately be the goal.

We use the following notation. For $A, B \subseteq V(G)$, $E(A, B)$ is the set of edges with one end in $A$ and the other in $B$. By $E(A)$ we mean $E(A, V(G) \setminus A)$ and instead of $E(\{a\}, B)$ we will write $E(a, B)$. Let $e(A, B) = |E(A, B)|$, and we will sometimes write $e(a, B)$ as $\deg(a, B)$. For a subgraph $H \subseteq G$, $e(a, H)$ means $e(a, V(H))$. Let $\Delta(A, B) := \max\{e(a, B) : a \in A\}$ and $\delta(A, B) := \min\{e(a, B) : a \in A\}$. We denote the graph induced by $A$ as $G[A]$. Given a tree $T$, we write $xTy$ for the unique path in $T$ between vertices $x$ and $y$. We will use the symbol $\oplus$ to denote modular addition, where the modulus will be clear in context.

2 Auxiliary facts

We begin with some facts that we will need throughout the paper.

Lemma 2.1. Let $G$ be a connected balanced $U, V$-bigraph on $2n$ vertices. Then $G$ contains a path of order $t = \min\{2(\delta_U + \delta_V), 2n\}$.

Proof. Let $P$ be any maximal path with $|P| < t$. It suffices to show that $G$ has a path $Q$ with $|Q| > |P|$. Since $P$ is maximal, the neighborhoods of the ends of $P$ are contained in $P$. We consider two cases depending on the parity of $P$. 3
Case 1: \( P = x_1y_1 \ldots x_l y_l \) is an even path. Then \( e(x_1, P) + e(y_l, P) \geq \delta_U + \delta_V > l \). Thus there exists an index \( i \in [l] \) such that \( x_1 \sim y_i \) and \( y_l \sim x_i \). So \( C = x_1 y_1 P y_l x_l P x_l \) is a cycle of length \( 2l \). Since \( t \leq 2n \) and \( G \) is connected, some vertex \( z \in P \) has a neighbor \( r \in G - C \). Then \( Q = rz(C-z) \) is a longer path.

Case 2: \( P = x_1 y_1 \ldots x_l y_l x_{l+1} \) is an odd path. Without loss of generality, let \( x_1 \in U \). Set \( P' = P - x_{l+1} \) and consider the components of \( G' = G - P' \). The component containing \( x_{l+1} \) has order 1 and thus more vertices from \( V \) than \( U \). Since \( G' \) is balanced it also has a component \( D \) with more vertices from \( V \) than \( U \). Since \( G \) is connected, there exists a vertex \( r \in D \) that is adjacent to a vertex \( z \in \{x_j, y_j\} \subseteq V(P') \). If possible, we choose \( r \in V \) and with respect to this condition, choose \( r \) so that \( j \) is maximized. Let \( w \) be the predecessor of \( z \) on \( P' \). If \( |D| = 1 \) then \( e(r, P') + e(x_1, P') \geq \delta_U + \delta_V > l \), so there exists an index \( i \in [l] \) such that \( x_1 \sim y_i \) and \( r \sim x_i \). Thus \( Q = r x_i P x_1 y_i P x_{l+1} \) is a path with \( |Q| > |P| \). So we may assume that \( |D| \geq 2 \). Fix a depth first search tree \( T \) of \( D \) that is rooted at \( r \). Let \( b \) be the number of leaves of \( T \) in \( V \). Note that

\[
2|T \cap V| - b \leq |E(T)| = |T| - 1 = |D \cap U| + |D \cap V| - 1
\]

which implies \( b \geq |D \cap V| - |D \cap U| + 1 \geq 2 \). Let \( y \) be a leaf of \( T \) in \( V \) that is distinct from \( r \). Since \( T \) is a depth first search tree, \( N(y) \subset V(yTrU \cap P') \). Let \( m = |V(yTrU) \cap P' \) and let \( i \) be the largest index with \( x_1 \sim y_i \). If \( j > l - m \) then \( Q = yTrz P x_1 \) is a path with \( |Q| = 2(j + m) \geq 2(l + 1) > |P| \). So suppose \( j \leq l - m \). If \( i > l - m \) then \( Q = yTrz P x_i P w \) is a path with \( |Q| \geq 2(i + m) \geq 2(l + 1) > |P| \). Otherwise \( i \leq l - m \). By choice of \( r \) we have \( e(x_1, P y_{l-m}) + e(y, P x_{l-m}) \geq \delta_U + \delta_V > m - l - m \). So there exists an index \( h \in [l - m] \) such that \( x_1 \sim y_h \) and \( y \sim x_h \). Thus \( Q = r Ty r x_h P x_1 y_h P x_{l+1} \) is a path with \( |Q| > |P| \).

Lemma 2.2. Let \( G \) be a balanced \( U, V \)-bigraph on \( 2n \) vertices.

(i) If \( e_s \) and \( e_t \) are independent edges and \( \delta(G) \geq \frac{3}{2} n + 1 \) then \( G \) contains a spanning ladder, starting with \( e_s \) and ending with \( e_t \).

(ii) If \( \Lambda = \{L^1, \ldots, L^s\} \) is a set of disjoint ladders in \( G \) such that \( \sum_{L \in \Lambda} |L| = 2t \) and \( \delta(G) \geq \frac{3}{2} n + s + t + 1 \) then \( G \) has a spanning ladder starting with the first rung \( e_1 \) of \( L^1 \), ending with the last rung \( e_2 \) of \( L^s \), and containing each \( L \in \Lambda \).

Proof. (i) Let \( M \) be a 1-factor of \( G \) with \( e_s, e_t \in M \). Define an auxiliary graph \( H = (M, F) \) on \( M \) as follows. If \( uw, xy \in M \) with \( u, x \in U \) then \( u \sim_H x y \) if and only if \( u \sim_G y \) and \( v \sim_G x \). There is a natural one-to-one correspondence between ladders \( u_1 v_1 \ldots u_h v_h \) in \( G \), whose rungs are in \( M \), and paths in \( H \). Also \( |H| = n \) and \( \delta(H) \geq \frac{1}{2} n + 1 \). So \( H \) is hamiltonian connected and thus has a Hamilton path, starting with \( e_s \) and ending with \( e_t \). This path corresponds to the required ladder in \( G \).

(ii) Note that \( \delta(G) \) is large enough to insure that \( G \) has a 1-factor \( M \) containing all the rungs of the ladders \( L^i \). Form \( H \) as in (i). Then each ladder \( L^i \) corresponds to a path \( P_i \) in \( H \) and \( \delta(H) \geq \frac{3}{2} n + s + t + 1 \). Thus any two vertices of \( H \) share \( s \) non-path neighbors. For \( i \in [s-1] \), connect the end \( c_i \) of each \( P_i \) to the start \( b_{i+1} \) of each \( P_{i+1} \) with a non-path vertex \( x_i \) to form a path \( P \subseteq H \) with \( |P| = t + s - 1 \). Let \( H' = H - (P - \{c_{s-1}, x_{s-1}\}) \). Then \( \delta(H') \geq \frac{1}{2} |H'| + 1 \) and so \( H' \) is hamiltonian connected. It follows that \( H' \) contains a Hamilton path \( Q \) starting at \( c_{s-1} \) and ending at \( x_{s-1} \). Then the Hamilton path \( b_1 P_c_{s-1} Q x_{s-1} P c_s \) of \( H \) corresponds to the required ladder in \( G \).
Observe that in the proof of Lemma 2.2(ii) we do not need the degrees of "interior" vertices of \( L^i \) to be large. More precisely, given a ladder \( L \) we define the partition \( V(L) = \text{ext}(L) \cup \tilde{L} \), where \( \text{ext}(L) \) is the set of exterior vertices, and \( \tilde{L} \) is the set of interior vertices. If \( L \) is an initial ladder, let \( \text{ext}(L) \) be the vertices in the last rung. If \( L \) is a terminal ladder, let \( \text{ext}(L) \) be the vertices in the first rung. If \( L \) is not an initial or terminal ladder, let \( \text{ext}(L) \) be the vertices in the first and last rung of \( L \). Note that if \( L \in \{L_1, L_2\} \), then it is possible for \( \tilde{L} = \emptyset \). Set \( I := I(\Lambda) = \bigcup_{L \in \Lambda} \tilde{L} \). Then Lemma 2.2(ii) still holds if we only require \( \deg(v) \geq \frac{3n + t + 4q}{4} + 1 \) for \( v \in V(G) \setminus I \).

**Lemma 2.3.** Let \( G \) be a balanced \( U, V \)-bigraph on \( 2n \) vertices and let \( \Lambda = \{L^1, \ldots, L^s\} \) be a set of disjoint ladders with initial ladder \( L^1 \) and if \( s > 1 \), terminal ladder \( L^s \) such that \( \sum_{L \in \Lambda} |L| = 2t \). Suppose \( \deg(v) \geq d \) for all \( v \notin I(\Lambda) \) and there exists \( Q \subseteq U \cup V \) with \( |Q| \leq q \) such that \( \deg(v) \geq D \) for every \( v \notin Q \cup I(\Lambda) \). If

\[
(i) \quad D \geq \frac{3n + 3s + t + 4q}{4} + 1 \quad \text{and} \quad (ii) \quad d > t + 3q + 2s + n - D.
\]

then \( G \) has a spanning ladder that starts with the first rung \( e_1 \) of \( L^1 \), and, if \( s > 1 \), ends with the last rung \( e_2 \) of \( L^s \).

**Proof.** Let \( M \) be a matching that saturates \( Q = Q \setminus I \) and avoids the ladders in \( \Lambda \). This is possible since \( q' = |Q'| \leq d - t \) by (ii). We view each edge of \( M \) as a 1-ladder. Let \( \Lambda^+ = \Lambda \cup M \), \( s' = s + q' \) and \( t' = t + q' \). Next we extend each ladder \( L \in \Lambda^+ \) to a new ladder \( \phi(L) \) as follows: let \( \phi(L^1) = L^1 y_1 z_1 \), \( \phi(L^i) = a_i b_i L^i \), and \( \phi(L^i) = a_i b_i L^i y_i z_i \) for \( i \in [s'] \setminus \{1, s\} \) such that \( a_h, b_h, y_h, z_h \notin R \cup R' \) for \( h \in [s'] \), where \( R = \bigcup_{L \in \Lambda^+} V(L) \) and \( R' \) is the set of all previously chosen extension vertices. For example, suppose we want to find \( y_{s'} z_{s'} \) after finding all previous extensions. Let \( u, v \in \text{ext}(L^{s'}) \) be the rung of \( L^{s'} \) that we wish to extend, where \( u, v \in \text{ext}(L^{s'}) \). We have \(|(R \cup R') \cap N(v)| < 2s' + t' \), and so it is possible by (ii) to choose \( y_{s'} \in N(v) \setminus (R \cup R') \). Note that \( Q \cup I(\Lambda) \subseteq R \), and so \( \deg(u) \geq D \). Now since \( D \leq n \) we have \( 3s + t + 4q + 4 \leq n \) and thus

\[
|\{(u,v) \in N(u) \cap N(v) : \phi(L) \in \Lambda^+ \}| - (R \cup R')| \geq \frac{1}{2} [n - (s + t + 2q)] + 2 \geq 1.
\]

So by (i) and (1) we may choose \( z_{s'} \in N(u) \cap N(y_{s'}) \setminus (R \cup R') \).

Set \( \Lambda' = \{\phi(L) : L \in \Lambda^+ \} \) and \( t'' = t' + 2s' - 2 \). Then \( s' = |\Lambda'| \) and \( 2t'' = \sum_{L \in \Lambda'} |L'| \). By (i)

\[
D \geq \frac{3n + 3s + t + 4q}{4} + 1 \geq \frac{3n + (s + q') + (t + q' + 2s + q')}{4} + 1 \geq \frac{3n + s' + t''}{4} + 1.
\]

Thus by Lemma (2.2), \( Q \subseteq R \subseteq I(\Lambda') \) and our observation preceding the Lemma, we are done. \( \square \)

### 3 Set-up and organization of the proof

For the rest of this section, let \( G \) and \( H \) be a balanced \( U, V \)-bigraphs on \( 2n \) vertices. Assume \( \delta_U + \delta_V \geq n + 2 \) and suppose without loss of generality that \( \delta_U \leq \delta_V \). Note that this implies \( \delta_U \geq 3 \). Define \( \gamma_1 \) by \( \delta_U = \gamma_1 n + 1 \) and \( \gamma_2 \) by \( \gamma_1 + \gamma_2 = 1 \). Assume \( \gamma_1 < \frac{1}{2} < \gamma_2 \), since the case where \( \gamma_1 = \gamma_2 \) was handled in [8]. Also assume \( \Delta(H) \leq 2 \) and \( k = \text{comp}(H) \). Our goal is to show that \( G \) contains \( H \).

The rest of the proof is organized in the following way. Our main task is to prove Theorem 1.8. This proof divides into three main cases. In Section 4 we handle the case that \( \gamma_1 < \frac{1}{2m} \). In this case,
we will show that $G$ contains $H$ for any value of $n$, but will not prove the existence of a spanning ladder. Otherwise, we consider two cases, the extremal case and the random case. The case is determined by whether $G$ is $\alpha$-splittable for a sufficiently small $\alpha$. In Section 5 we define $G$ to be $\alpha$-splittable if a certain configuration exists in $G$. The definition is designed to be most useful in the random case where $G$ fails to be $\alpha$-splittable. In the remainder of Section 5 we show that if $G$ is $\alpha$-splittable and $\beta \geq 2\sqrt{\alpha}$ then $G$ has a much nicer configuration called a $\beta$-partition. In Section 6, we handle the extremal case by showing that for sufficiently small $\beta$, we can obtain a spanning ladder from any $\beta$-partition. In Section 7 we introduce the Regularity and Blow-up Lemmas. In Section 8 we use these lemmas to prove that in the random case, if $n$ is sufficiently large in terms of $\alpha$, then $G$ contains a spanning ladder. In Section 9 we use our previous results to complete the proofs of Theorem 1.9 and Theorem 1.10.

4 Pre-extremal Case

In this section, we will show that Theorem 1.8 is true in the case that one of the minimum degrees is very small.

Lemma 4.1. If $\gamma_1 < \frac{1}{200k}$ then $G$ contains $H$.

Proof. Let $S = \{u \in U : \deg(u) < \frac{n}{10} \}$ and $s = |S|$. Then $\gamma_2 > 1 - \frac{1}{200k}$ and

$$\left(1 - \frac{1}{200k}\right)n^2 \leq \sum_{v \in V} \deg(v) = \sum_{u \in U} \deg(u) < \frac{9}{10}ns + n(n - s)$$

$$s < \frac{1}{20k}n. \quad (2)$$

Since $\delta_U + \delta_V \geq n + 2$, $G$ contains a Hamilton cycle $D$. Suppose $D$ orders $S$ as $x_1, \ldots, x_s$, where $x_1$ is chosen so that $\text{dist}_D(x_1, x_s) > 2$. For each $i \in [s]$, let $w_i x_i y_i \subseteq D$. Since

$$|\{N(w_i) \cap N(y_i) \} \setminus S| \geq \left(1 - \frac{1}{100k} - \frac{1}{20k}\right)n > s,$$

we can choose distinct $z_i \in U$ such that $z_i$ is adjacent to both $y_i$ and $w_i x_i$, if $y_i = w_i x_i$ then $z_i = x_i$, and otherwise $z_i \notin S$. Note that by the choice of $x_1$ we have $y_i \neq w_1$ and thus $z_i \neq x_1$. Set $C = w_1 x_1 y_1 z_1 \ldots w_s x_s y_s z_s w_1$. Then $C$ is a cycle with length at most $4s < \frac{2n}{k}$. Let $G' = G - (C - \{w_1, z_s\})$. Then $G'$ is a balanced bipartite graph and $G' \subseteq G - S$. Thus

$$\delta(G') \geq \frac{9}{10}n - 2s \geq \frac{3}{4}n + 1 \geq \frac{3 |G'|^2}{4 \cdot 2} + 1.$$

So by Lemma 2.2(1), $G'$ contains a spanning ladder $L$ with first rung $w_1 z_s$. Since $\text{comp}(H) = k$, some component of $H$ must have size at least $\frac{2n}{k}$ and thus $H \subseteq C \cup L \subseteq G$. \hfill $\square$

5 Splitting

In this section we define the notions of $\alpha$-splitting and $\beta$-partition. We prove that if $G$ has an $\alpha$-splitting then it has a $\beta$-partition.
Definition 5.1. G is α-splittable with α-splitting \((X,Y)\) if \(X \subseteq U\) and \(Y \subseteq V\) satisfy

\[(1) \quad (\gamma_1 - \alpha)n \leq |X| \leq (\gamma_1 + \alpha)n \quad \text{and} \quad (\gamma_2 - \alpha)n \leq |Y| \leq (\gamma_2 + \alpha)n \quad \text{and}
\]

\[(2) \quad e(X,Y) \leq \alpha |X||Y|
\]

Informally, the following lemma asserts that if \(G\) is \(\alpha\)-splittable then \(G\) can almost be split into two balanced complete bipartite graphs so that one has order approximately \(2\gamma_1 n\) and the other has order approximately \(2\gamma_2 n\). Let \((X,Y)\) be an \(\alpha\)-splitting of \(G\) and set \(X = U \setminus X\) and \(Y = V \setminus Y\).

Lemma 5.2. If \(G\) is \(\alpha\)-splittable for \(\alpha \leq (\frac{2}{3})^2\), then there exist partitions \(U = X_0 \cup X_1 \cup X_2\) and \(V = Y_0 \cup Y_1 \cup Y_2\) so that

\[(1) \quad X_1 \subseteq X, Y_1 \subseteq Y, |X_1| = |Y_1| \geq (\gamma_1 - 2\sqrt{\alpha})n \quad \text{and} \quad \delta(G[X_1 \cup Y_1]) \geq (\gamma_1 - 4\sqrt{\alpha})n \quad \text{and}
\]

\[(2) \quad X_2 \subseteq \overline{X}, Y_2 \subseteq Y, |X_2| = |Y_2| \geq (\gamma_2 - 2\sqrt{\alpha})n \quad \text{and} \quad \delta(G[X_2 \cup Y_2]) \geq (\gamma_2 - 4\sqrt{\alpha})n.
\]

Proof. We will show that there exist \(X_1 \subseteq X\) and \(Y_1 \subseteq Y\) satisfying (i) without using \(\gamma_1 < \gamma_2\). Then by the symmetry of \(\gamma_1, X\) and \(\gamma_2, Y\) it will follow that there exists \(Y_2 \subseteq Y\) and \(X_2 \subseteq \overline{X}\) satisfying (ii).

Let \(S = \{x \in X : e(x,Y) < (\gamma_1 - \sqrt{\alpha})n\}\). Then

\[|S|\sqrt{\alpha} n \leq \sum_{x \in X} e(x,Y) = e(X,Y) \leq \alpha |X||Y|
\]

\[|S| \leq \sqrt{\alpha} |X| |Y| \leq \sqrt{\alpha} n.
\]

Let \(T = \{y \in Y : e(y,X) < (\gamma_1 - \sqrt{\alpha})n\}\). Then since \(\sum_{x \in X} e(x,Y) = e(X,Y) = \sum_{y \in Y} e(y,X)\), we have

\[\gamma_1 n |X| - \alpha |X||Y| \leq e(X,Y) \leq (\gamma_1 - \sqrt{\alpha})n |T| + |X|(|Y| - |T|).
\]

Thus

\[|X| - (\gamma_1 - \sqrt{\alpha})n |T| \leq (|Y| - \gamma_1 n + \alpha |Y|) |X|
\]

\[(\sqrt{\alpha} - \alpha) |T| \leq ((\gamma_1 + \alpha - \gamma_1 n + \alpha (\gamma_2 + \alpha)n)(\gamma_1 + \alpha)n
\]

\[1 - \sqrt{\alpha} |T| \leq (1 + \gamma_2 + \alpha)(\gamma_1 + \alpha)\sqrt{\alpha} n
\]

\[|T| \leq \frac{3}{2} \sqrt{\alpha} n.
\]

Choose \(X_1 \subseteq X - S\) and \(Y_1 \subseteq Y - T\) such that \(|X_1| = |Y_1| \geq (\gamma_1 - 2\sqrt{\alpha})\). This is possible by Definition 5.1(i) and the upper bounds (3) and (4) on \(|S|\) and \(|T|\). Thus for every \(x \in X_1, y \in Y_1\)

\[e(x,Y_1) \geq e(x,Y) - |T| \geq ((\gamma_1 - \sqrt{\alpha}) - 2\sqrt{\alpha})n \geq (\gamma_1 - 4\sqrt{\alpha})n \quad \text{and}
\]

\[e(y,X_1) \geq e(y,X) - |S| \geq ((\gamma_1 - \sqrt{\alpha}) - 2\sqrt{\alpha})n \geq (\gamma_1 - 4\sqrt{\alpha})n.
\]

\[\square\]

Definition 5.3. A \(\beta\)-partition of \(G\) is an ordered partition \((X_1, S_1, S_2, X_2, Y_1, T_1, T_2, Y_2)\) with \(U = U_1 \cup U_2, U_1 = X_1 \cup S_1, U_2 = S_2 \cup X_2, V = V_1 \cup V_2, V_1 = Y_1 \cup T_1, V_2 = T_2 \cup Y_2\) such that for \(g := ||S_1| - |T_1||\) and \(h \in [2]\) the following conditions are satisfied

\[7\]
(i) \((\gamma_h - \beta)n \leq |U_h|, |V_h| \leq (\gamma_h + \beta)n\);
(ii) \(|S_1|, |S_2|, |T_1|, |T_2| \leq 2\beta n;\)
(iii) \(\delta(X_h, Y_h), \delta(Y_h, X_h) \geq (\gamma_h - 4\beta)n + g;\)
(iv) \(\delta(S_h, Y_h), \delta(T_h, X_h) \geq 2\beta n + g;\)
(v) if \(|S_i| > |T_i|\) then \(\Delta(U_i, V_j), \Delta(V_j, U_i) < 24\beta n\) for \(i \in [2]\) and \(j = 3 - i.\)

**Lemma 5.4.** If \(G\) is \(\alpha\)-splittable and \(2\sqrt{\alpha} \leq \beta \leq \frac{2\alpha}{38}\) then \(G\) has a \(\beta\) partition.

**Proof.** (See Fig. 1.) We start with the partition \(U = X_0 \cup X_1 \cup X_2\) and \(V = Y_0 \cup Y_1 \cup Y_2\) from Lemma 5.2. We describe a process for updating the partition so that conditions (i-v) are satisfied.

Set
\[ S_1 = \{ x \in X_0 : e(x, Y_1) \geq 24\beta n \}, S_2 = X_0 \setminus S_1, T_1 = \{ y \in Y_0 : e(y, X_1) \geq 24\beta n \} \text{ and } T_2 = Y_0 \setminus T_1. \]

Clearly (i,ii) hold. Also (iii) holds with \(2\beta n - g\) to spare. Since \(50\beta \leq \gamma_1 \leq \gamma_2\), we have \(e(x, Y_2), e(y, X_2) \geq 24\beta n\) for all \(x \in S_2\) and \(y \in T_2\), and thus (iv) also holds with \(2\beta n - g\) to spare. If (v) holds, we are done, so suppose not. Choose \(i\) such that \(|S_i| > |T_i|\) and set \(j = 3 - i\), then \(0 < g_0 := |S_i| - |T_i| = |T_j| - |S_j| \leq 2\beta n\). We will now move vertices so that after each move, the difference \(|S_i| - |T_i|\) is reduced while (i-iv) continue to hold. Once the difference can no longer be reduced by moving vertices we will claim that (v) holds and then we set \(g := |S_i| - |T_i| \geq 0.\) On each step we attempt to move vertices \(x \in S_i\) with \(e(x, Y_j) \geq 24\beta n\) from \(S_i\) to \(S_j\) and/or vertices \(y \in T_j\) with \(e(y, X_i) \geq 24\beta n\) from \(T_j\) to \(T_i\). If no vertices meet this requirement, then we will attempt to move vertices \(x \in X_i\) with \(e(x, Y_j) \geq 24\beta n\) from \(X_i\) to \(T_j\). Any time a move of this type is made the size of \(X_i\) is reduced, so to ensure that \(|X_h| = |Y_h|\) we must also move any vertex from \(Y_i\) to \(T_i\). Similarly, we may move eligible vertices from \(Y_j\) to \(T_i\) and compensate by moving any vertex from \(X_j\) to \(S_j\). After each move, any of \(|X_h|, |Y_h|, \delta(X_i, Y_i), \delta(Y_i, X_i), \delta(S_i, Y_i), \delta(T_i, X_i)\) may decrease, and \(|S_j|\) and \(|T_j|\) will increase. Note that these parameters may change by only 1 per move. Since we will make at most \(g_0 - g\) moves, (iii,iv) will continue to hold. Furthermore, since \(|S_i|, |T_j|\) will never be increased, \(|U_i|, |V_j|\) may decrease by at most \(g_0 - g\) and \(|U_j|, |V_i|\) may increase by at most \(g_0 - g\), so (i,ii) will continue to hold. When the the process stops, (v) will hold either because \(|S_i| = |T_i|\) or because there are no more eligible vertices to move, in which case condition (v) is satisfied.
6 Extremal case

In this section we prove Theorem 1.8 in the case that \( G \) is \( \alpha \)-splittable for sufficiently small \( \alpha \).

Lemma 6.1. Let \( N_1(k) = 408800k + 1 \). If \( n \geq N_1(k) \), \( \gamma_1 \geq \frac{1}{200k} \), and \( G \) is \( \alpha \)-splittable for \( \alpha = \left( \frac{\gamma_1 n}{200} \right)^2 \), then \( G \) contains a spanning ladder.

Proof. Set \( \beta = 2\sqrt{\alpha} = \frac{\gamma_1 n}{200} \), then by Lemma 5.4 \( G \) has a \( \beta \)-partition \( (X_1, S_1, S_2, X_2, Y_1, T_1, T_2, Y_2) \). Since \( \gamma_1 \geq \frac{4}{200k} \) we have
\[
\beta n = \frac{\gamma_1 n}{200} > 7. \tag{5}
\]

Set \( G_i = G[U_i \cup V_i] \) for \( i \in [2] \). For \( L \in \{ L_2, L_3 \} \) we say that \( L \) is a crossing ladder if its first rung is in \( G_1 \) and its last rung is in \( G_2 \). Choose \( i \) so that \( g = |S_i| - |T_i| \geq 0 \) and set \( j = 3 - i \). Roughly, our plan is to find a crossing ladder \( L^0 \) and then find ladders \( L', L'' \) spanning \( G_1, G_2 \) such that the last rung of \( L' \) is the first rung of \( L^0 \) and the last rung of \( L'' \) is the first rung of \( L'' \). However \( G_1, G_2 \) may not be balanced or \( G_1, G_2 \) may have been balanced to begin with, but the crossing ladder created an imbalance. In both of these situations we will need a way of moving vertices between \( G_1 \) and \( G_2 \) so that they may be incorporated into \( L' \) and \( L'' \).

Formally, our plan is to construct a set of pairwise disjoint ladders \( \Lambda = \{ L^0, \ldots, L^s \} \) with \( s \leq g + 1 \leq 2\beta n + 1 \) and \( I = I(\Lambda) = \bigcup_{L \in \Lambda} L \) such that
(a) \( L^0 \) is a crossing ladder,
(b) for all \( p \in [s] \), there exists \( h \in [2] \) with \( \text{ext}(L_p) \subseteq G_h \) and
(c) \( G_1 - I \) is balanced (equivalently, \( G_2 - I \) is balanced).

We may also designate one ladder as an initial ladder for each \( G_h \). Then we will apply Lemma 2.3 to construct a spanning ladder.

We begin with two useful facts. By our degree conditions we have
\[
\forall v, v' \in V \quad |N(v) \cap N(v')| \geq 2\delta_V - n > 2(n/2 + 1) - n = 2 \tag{6}
\]
Since \( \sum_{u \in U} \deg(u) = e(U, V) \geq \delta_V |U| \) and \( \delta_U < \delta_V \), there exists \( u^* \in U \) with \( \deg(u^*) > \delta_V \).

Thus
\[
\exists u^* \in U \forall u \in U \quad |N(u^*) \cap N(u)| \geq \delta_V + 1 + \delta_U - n \geq 3. \tag{7}
\]

Step 1: (Construct a crossing ladder \( L^0 \).) We are done unless
\[
\text{there is no crossing } L_2. \tag{*}
\]
So suppose not, then by (7) there exist vertices \( x_1 \in U_1, x_2 \in U_2 \) such that \( |N(x_1) \cap N(x_2)| \geq 3 \) and
\[
(N(x_1) \cap N(x_2) \subseteq V_1) \cup (N(x_1) \cap N(x_2) \subseteq V_2). \tag{+1}
\]

Let \( y_1, y_2 \in N(x_1) \cap N(x_2) \), by (1) there exists \( q \in [2] \) such that \( \{y_1, y_2\} \subseteq V_q \). Let \( q' = 3 - q \) and \( y_3 \in N(x_{q'}) \cap V_{q'} \). By (6), \( y_2 \) and \( y_3 \) have a common neighbor \( x_3 \neq x_q, x_{q'} \). By (1), \( x_3 \in U_{q'} \). Thus \( L^0 = x_qy_1y_q'x_{q'}y_3x_3y_3 \) is a crossing \( L_3 \). (See Fig. 2)
Step 2: (Construct $L^1, \ldots, L^s$ so that (b) and (c) hold.) For all $u \in U_i$ and $v \in V_j$

$$n + 2 \leq \deg(u) + \deg(v) \leq |V_i| + e(u, V_j) + |U_j| + e(v, U_i) \leq n - g + e(u, V_j) + e(v, U_i).$$

Therefore

$$g + 2 \leq \delta(U_i, V_j) + \delta(V_j, U_i). \quad (8)$$

Case 1: $g = 0$. If $G$ has a crossing $L_2$, i.e., $(\ast)$ fails, then there is nothing to do. Otherwise, $L^0 = L_3$ and $y_2 \in L^0 \cap V_q$ thus $|U_q \cap L^0| = |V_q \cap L^0| + 1$. Let $x' \in N(y_2) \cap (U_q - x_q)$ and $y' \in N(x_q) \cap (V_q - y_3)$. Since $g = 0$, $i$ and $j$ are interchangeable, so by (8), either $x'$ has a neighbor in $V_{q'}$ or $y'$ has a neighbor in $U_q$ and by $(\ast)$, neither of these possible neighbors can be in $L^0$. Regardless, there exists an edge $xy \in E(U_q, V_{q'})$ whose ends are not in $L^0$. Let $y' \in N(x) \cap (V_q \setminus V(L^0))$. By (6), $y$ and $y'$ have a common neighbor $x^*$ with $x^* \neq x, x_h$. By $(\ast)$, $x^* \in U_q$. Set $L^1 = xyx'y^*$ and specify $L^1$ as the initial ladder for $G_q$. Note that $\operatorname{ext}(L^1) \subseteq G_q$ and $|U_q \setminus (L^0 \cup L^1)| = |V_q \setminus (L^0 \cup L^1)|$ so we are done.

Case 2: $g \geq 1$. Using Definition 5.3(i,v) and $g \geq 1$ we have

$$\forall v, v^* \in V_j \ |(N(v) \cap N(v^*)) \cap U_j| \geq 2(\gamma_2 - 2\beta)n - |U_j| \geq |U_j| - 50\beta n > \frac{4}{5}|U_j|. \quad (9)$$

If $U_i = U_1$ we have

$$\forall u, u^* \in U_1 \ |(N(u) \cap N(u^*)) \cap V_i| \geq 2(\gamma_1 - 2\beta)n - |V_i| \geq |V_i| - 50\beta n > \frac{4}{5}|V_i|. \quad (10)$$

If $U_i = U_2$ then for all $v \in V_i$, $(\gamma_1 + \beta)n \geq \deg(v, U_1) \geq (\gamma_2 - 2\beta)n$ which implies $\gamma_2 > \gamma_1 \geq \gamma_2 - 25\beta$. In which case we have

$$\forall u, u^* \in U_2 \ |(N(u) \cap N(u^*)) \cap V_2| \geq 2(\gamma_1 - 2\beta)n - |V_2| \geq 2(\gamma_2 - 49\beta)n - |V_2| \geq |V_2| - 100\beta n > \frac{13}{20}|V_2|. \quad (11)$$

Let $m = \max\{\delta(U_i, V_j), \delta(V_j, U_i)\}$ and note that by (8) and $g \geq 1$, we have $m \geq 2$. Also note that by (8), if $g \geq 3$ then $m \geq 3$. It is the case that if $L^0 = L_3$ then $m \geq 3$: if $\delta(V_j, U_i) > 0$, then by (6,\ast), we have $\delta(V_j, U_i) \geq 3$ otherwise $\delta(V_j, U_i) = 0$ and thus $\delta(U_i, V_j) \geq 3$ by (8).

Case 2a: $m = 2$. Then $L^0 = L_2$, $1 \leq g \leq 2$ and $1 \leq \delta(A, B) \leq \delta(B, A) = 2$ for some choice of $\{A, B\} = \{U_i, V_j\}$. Let $A \cup A', B \cup B' \in \{U, V\}$. By Definition 5.3(v) and $g > 0$ there exists
b₁ ∈ B \ V(L⁰) with no neighbor in V(L⁰) ∩ A and two neighbors a₁, a₂ ∈ A. By (9,10,11), a₁ and
a₂ have a common neighbor b₂ ∈ B' \ V(L⁰). Let L¹ = a₁b₁a₂b₂ be the initial ladder for Gₜ, where
b₂ ∈ Gₜ and ext(L¹) ⊆ Gₜ. If g = 1 then |U₁ \ (L⁰ ∪ L¹)| = |V₁ \ (L⁰ ∪ L¹)| and we are done. If
|L¹| = 2 then also δ(A, B) = 2 by (8), and a similar argument yields an initial ladder L² = a₃b₃a₄b₄
for Gₜ₋₁ such that a₃ ∈ A, b₃, b₄ ∈ B, a₄ ∈ A' and L⁰, L¹, L² are disjoint. We have ext(L²) ⊆ Gₜ₋₁
and |U₁ \ (L⁰ ∪ L¹ ∪ L²)| = |V₁ \ (L⁰ ∪ L¹ ∪ L²)| so we are done.

**Case 2b:** m ≥ 3. By (8) there exists A ∈ \{Uᵣ, Vᵣ\} = \{A, B\} such that e(a, B) ≥ m ≥ 3 for all
a ∈ A. Let M = \{aᵣ, bᵣ, cᵣ, dᵣ : r ∈ [s]\} be a maximal set of disjoint claws with root aᵣ ∈ A
and leaves bᵣ, cᵣ, dᵣ ∈ B. Then every vertex in \bar{A} = A \ \{aᵣ : r ∈ [s]\} has at least m − 2 neighbors in
N = \{bᵣ, cᵣ, dᵣ : r ∈ [s]\}. Suppose s ≤ g. Then using Definition 5.3(i,v), g ≤ 2βn and g ≤ 2m − 2
(from (8)), we note

\[(m − 2)((γ₁ − β)(n − s)) ≤ |E(\bar{A}, N)| ≤ 3s \cdot 24βn.\]

Thus
\[γ₁ ≤ 72β \cdot \frac{g}{m − 2} + \frac{s}{n} ≤ 72β \cdot \frac{2m − 2}{m − 2} + 3β ≤ 291β < γ₁,\]

a contradiction. So we conclude that s ≥ g + 1. Choose B' so that \{B, B'\} = \{Uᵣ, Vᵣ\} for some
r ∈ [2]. Let g' := \|B \cap L⁰| − |B' ∩ L⁰| and note that g − 1 ≤ g' ≤ g + 1. In order to balance Gₜ − L⁰
we build a set of disjoint 3-ladders

Λ(M) = \{aᵣ, bᵣ, cᵣ, dᵣ : r ∈ \{g'\}, aᵣ, bᵣ, cᵣ, dᵣ \in M and xᵣ, yᵣ \in B'\}.

This is possible by s ≥ g + 1, (9,10,11) and

\[|(N(bᵣ) ∩ N(cᵣ)) \cap (B' \cap L⁰)|, |(N(cᵣ) ∩ N(dᵣ)) \cap (B' \cap L⁰)| ≥ \frac{13}{20}|B'| − 2 ≥ 2g'.\]

Thus |Uᵣ \ (L⁰ ∪ I(Λ(M)))| = |Vᵣ \ (L⁰ ∪ I(Λ(M)))| and ext(L) ⊆ Gₜ for all L ∈ Λ(M) so we are
done.

**Step 3:** (Construct the spanning ladder.) Let Λ be the set of ladders constructed in Steps 1 and
2 and set I := I(Λ). Let Λₜ = \{L ∈ Λ : ext(L) ⊆ Gₜ\} and Gₜ = (Gₜ − I) ∪ \bigcup Λₜ for h ∈ [2]. Note
that G₁', G₂' are balanced and G₁' ∩ G₂' = G = L⁰. For each ladder L ∈ Λₜ there is a unique vertex
v' ∈ L ∩ V(Gₜ₂). Since v' ∈ L, we are unconcerned about its degree in Gₜ so we add this vertex to
the appropriate exceptional set (Sₜ or Tₜ) in Gₜ'.

Let e₁ and e₂ be the first and last rungs of L⁰, which we will specify as the terminal ladders in
G₁' and G₂' respectively. It will suffice to show using Lemma 2.3 that each Gₜ' has a spanning
ladder, starting at its initial ladder, if it is specified in Case 1 or Case 2a, and ending at its terminal
ladder. Let s' := |Λₜ| ≤ g + 1 and t' := \frac{1}{3} \bigcup Λₜ ≤ 3(g + 1). Recall that g = |Sₜ| − |Tₜ|. Since
we only add vertices to Sₜ and Tₜ and L₀ ∩ V(Gₜ') = \emptyset, we have \n' := \frac{1}{3} |Gₜ'| ≤ (γₜ − 4β)n. Let
Q := \{v ∈ V(Gₜ') : \deg(v) < D\}, where D := (γₜ − 4β)n − 1. By Definition 5.3(iii), Q ⊆ Sₜ ∪ Tₜ.
Thus, by Definition 5.3(ii), q' := |Q| ≤ 4βn − g. By Definition 5.3(iii,iv), if v ∈ V(Gₜ') \ I then
d := 22βn − 1 ≤ 22βn + g − s' ≤ \deg(v, Gₜ'). Thus Gₜ' has the desired spanning ladder by Lemma
2.3, since

\[\frac{3n' + 3s' + t' + 4q'}{4} + 1 ≤ \frac{3γₜn + 23βn + 10}{4} ≤ D \quad \text{and} \quad t' + 3q' + 2s' + n' − D ≤ 21βn + 6 < d.\]


7 The Regularity and Blow-up Lemmas

In this section we review the Regularity and Blow-up Lemmas. Let \( \Gamma \) be a simple graph on \( n \) vertices. For two disjoint, nonempty subsets \( U \) and \( V \) of \( V(\Gamma) \), define the density of the pair \((U, V)\) as

\[
d(U, V) = \frac{e(U, V)}{|U||V|}.
\]

**Definition 7.1.** A pair \((U, V)\) is called \( \epsilon \)-regular if for every \( U' \subseteq U \) with \( |U'| \geq \epsilon |U| \) and every \( V' \subseteq V \) with \( |V'| \geq \epsilon |V| \), \( |d(U', V') - d(U, V)| \leq \epsilon \). The pair \((U, V)\) is \((\epsilon, \delta)\)-super-regular if it is \( \epsilon \)-regular and for all \( u \in U \), \( \deg(u, V) \geq \delta |V| \) and for all \( v \in V \), \( \deg(v, U) \geq \delta |U| \).

First we note the following facts that we will need.

**Lemma 7.2.** If \((U, V)\) is an \( \epsilon \)-regular pair with density \( \delta \), then for any \( Y \subseteq V \) with \( |Y| \geq \epsilon |V| \) there are less than \( \epsilon |U| \) vertices \( u \in U \) such that \( \deg(u, Y) < (\delta - \epsilon)|Y| \).

**Proposition 7.3.** If \((U, V)\) is a balanced \( \epsilon \)-regular pair with density \( \delta \geq 2\sqrt{\epsilon} > 0 \) and subsets \( A, C \subseteq U \), \( B, D \subseteq V \) of size at least \( \frac{1}{2}\epsilon |U| \) then there exist \( a \in A, b \in B, c \in C, d \in D \) with \( abcd a = C_4 \).

**Lemma 7.4** (Slicing Lemma). Let \((U, V)\) be an \( \epsilon \)-regular pair with density \( \delta \), and for some \( \lambda > \epsilon \) let \( U' \subseteq U \), \( V' \subseteq V \), with \( |U'| \geq \lambda |U| \), \(|V'| \geq \lambda |V| \). Then \((U', V')\) is an \( \epsilon' \)-regular pair of density \( \delta' \) where \( \epsilon' = \max\{\frac{\epsilon}{10}, 2\epsilon\} \) and \( \delta' \geq \delta - \epsilon \).

**Lemma 7.5** (Augmenting Lemma). Let \((U, V)\) be an \( \epsilon \)-regular pair. Suppose that \( U' = U \cup S \) and \( V' = V \cup T \), where \( |S| \leq \mu|U| \), \( |T| \leq \mu|V| \), \( S \cap V' = \emptyset = T \cap U' \), and \( 0 < \mu < \epsilon \). Then \((U', V')\) is an \( \epsilon' \)-regular pair, where \( \epsilon' = \max\{\frac{\mu}{10}, 6\epsilon\} \).

**Definition 7.6.** A partition \( \{V_0, V_1, \ldots, V_t\} \) of \( V(\Gamma) \) is called \( \epsilon \)-regular if the following conditions are satisfied:

(i) \( |V_0| \leq \epsilon |V| \).

(ii) For all \( i, j \in [t] \), \( |V_i| = |V_j| \).

(iii) All but at most \( \epsilon t^2 \) of pairs \((V_i, V_j)\), \( 1 \leq i, j \leq t \), are \( \epsilon \)-regular.

The parts of the partition are called clusters. Note that the cluster \( V_0 \) plays a distinguished role in the above definitions and is usually called the exceptional cluster (or class). Our main tool in the proof will be the Regularity Lemma of Szemerédi [19] which asserts that for every \( \epsilon > 0 \) every graph which is large enough admits an \( \epsilon \)-regular partition into a bounded number of clusters.

**Lemma 7.7** (Regularity Lemma). For every \( \epsilon > 0 \) there exists \( N := N(\epsilon, m) \) and \( M := M(\epsilon, m) \) such that every graph on at least \( N \) vertices admits an \( \epsilon \)-regular partition \( \{V_0, V_1, \ldots, V_t\} \) with \( m \leq t \leq M \).

In the next section we will want a regular partition of a bipartite graph so we will use the following formulation (see for example [8]).
**Corollary 7.8** (Regularity Lemma - Bipartite Case). For every $\epsilon > 0$ there exists $N := N(\epsilon)$ and $M := M(\epsilon)$ such that every balanced $U, V$-bigraph on at least $2N^2$ vertices admits an $\epsilon$-regular partition $\{U_0, U_1, \ldots, U_t\} \cup \{V_0, V_1, \ldots, V_t\}$ with $t \leq M$ satisfying

(i) $|U_0| = |V_0| \leq \epsilon n$,

(ii) for all $i, j \in [t]$, $(1 - \epsilon)^n \leq |U_i| = |V_j| \leq \frac{\epsilon}{t}$ and

(iii) for all $U_i \in \{U_1, \ldots, U_t\}$ there are at most $\epsilon t$ sets $V_j \in \{V_1, \ldots, V_t\}$ such that $(U_i, V_j)$ is not $\epsilon$-regular and for all $V_i \in \{V_1, \ldots, V_t\}$ there are at most $\epsilon t$ sets $U_j \in \{U_1, \ldots, U_t\}$ such that $(V_i, U_j)$ is not $\epsilon$-regular.

In addition, we shall use the following version of the Blow-up Lemma [12].

**Lemma 7.9** (Blow-up Lemma). Given $\delta > 0$, $\Delta > 0$ and $\rho > 0$ there exist $\epsilon > 0$ and $\eta > 0$ such that the following holds. Let $S = (W_1, W_2)$ be an $(\epsilon, \delta)$-super-regular pair with $|W_1| = n_1$ and $|W_2| = n_2$. If $T$ is a $A_1, A_2$-bigraph with maximum degree $\Delta(T) < \Delta$ and $T$ is embeddable into the complete bipartite graph $K_{n_1, n_2}$ then it is also embeddable into $S$. Moreover, for all $n_1$, subsets $A'_i \subseteq A_i$ and functions $f_i : A'_i \rightarrow (\varphi)$, $i = 1, 2$, $T$ can be embedded into $S$ so that the image of each $a_i \in A'_i$ is in the set $f_i(a_i)$.

### 8 Random case

In this section, we will show that if the graph is not $\alpha$-splittable for sufficiently small $\alpha$ then it contains a spanning ladder. The proof is based on the Regularity Lemma of Szemerédi and the Blow-up Lemma of Komlos, Sárközy, and Szemerédi.

**Lemma 8.1.** Let $k$ be a positive integer and suppose $\gamma_1 \geq \frac{1}{200k}$. There exists an $N_2(k)$ so that if $G$ is not $\alpha$-splittable for $\alpha = \left(\frac{\gamma_1}{551}\right)^2$, and $n \geq N_2(k)$ then $G$ contains a spanning ladder.

**Proof.** Let $0 < d_0 \leq \frac{\alpha\gamma_12^2}{8}$, $\delta_1 \leq \frac{1}{3000}d_0^2$, $\delta_2 \leq \frac{1}{2}\delta_1$, $\delta_3 \leq \frac{1}{2}\delta_2$, $\delta_4 \leq \frac{1}{2}\delta_3$, $\delta \leq \frac{1}{2}\delta_4$, $\Delta = 4$ and $\rho = \frac{\epsilon}{\Delta^2}$. For these choices of $\delta, \Delta$ and $\rho$ choose $\epsilon < \delta^2$ and $\eta$ to satisfy the conclusion of Lemma 7.9. Now let $\epsilon_5 \leq \left(\frac{\epsilon}{\Delta}\right)^4$, $\epsilon_4 \leq \frac{1}{2}\epsilon_5$, $\epsilon_3 \leq \frac{1}{2}\epsilon_4$, $\epsilon_2 \leq \frac{1}{2}\epsilon_3$, and $\epsilon_1 \leq \frac{1}{2}\epsilon_2$. So

$$0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \epsilon_4 < \epsilon_5 \ll \epsilon \ll \delta < \delta_4 < \delta_3 < \delta_2 < \delta_1 \ll d_0 \ll \alpha.$$ 

Let $N_2(k) = \max\{N(\epsilon_1), \frac{4M(\epsilon_1)}{n}\}$, where $M(\epsilon_1)$ and $N(\epsilon_1)$ are the values obtained from Corollary 7.8. Apply Corollary 7.8 to $G$ with $\epsilon_1$ to obtain a partition $\{U_0, U_1, \ldots, U_t\} \cup \{V_0, V_1, \ldots, V_t\}$ satisfying (i-iii). For all $i, j \in [t]$, let $\ell := |U_i| = |V_j|$ and note that

$$(1 - \epsilon_1)^\frac{n}{\ell} \leq \ell \leq \frac{n}{\ell}.$$ 

Consider the cluster graph $G$ with $V(G) = \{U_1, \ldots, U_t\} \cup \{V_1, \ldots, V_t\}$ and two clusters $W, W'$ joined by an edge when the pair $(W, W')$ is $\epsilon_1$-regular and $d(W, W') \geq \delta_1$. Then $G$ is a bipartite graph with bipartition $\{U, V\}$, where $U = \{U_1, \ldots, U_t\}$ and $V = \{V_1, \ldots, V_t\}$. For a cluster $Z$, let $I(Z) = \{W : (Z, W) \text{ is irregular}\}$ and $\overline{I}(Z) = \{W : d(Z, W) < \delta_1\}$.

**Claim 8.2.** $G$ contains a path $P$ on $2q$ vertices with $q \geq (1 - 2\delta_1 - 4\epsilon_1)t$. 

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Proof. First note that
\begin{equation}
\delta_U \geq (\gamma_1 - \delta_1 - 2\epsilon_1)t \quad \text{and} \quad \delta_V \geq (\gamma_2 - \delta_1 - 2\epsilon_1)t. \tag{13}
\end{equation}

Otherwise there exists \( Z \in V(G) \) with \( \deg_G(Z) < (\gamma_1 - \delta_1 - 2\epsilon_1)t \), where \( i = 1 \) if \( Z \in U \) and \( i = 2 \) if \( Z \in V \). Then we have the following contradiction:
\[
\gamma_i n\ell \leq e_G(Z) \leq \sum_{W \in N_G(Z)} e(Z, W) + \sum_{W \in I(Z)} e(Z, W) + e(Z, V_0) < (\gamma_i - \delta_1 - 2\epsilon_1)t\ell^2 + \delta_1 t\ell^2 + \epsilon_1 t\ell^2 + \epsilon_1 n\ell \leq \gamma_i n\ell.
\]

Now suppose that \( G \) is disconnected, we will obtain a contradiction by showing that this implies that \( G \) is \( \alpha \)-splittable. Let \( A \) and \( B \) be distinct components of \( G \) and let \( X = U \cap \bigcup A \) and \( Y = V \cap \bigcup B \). Using \( e_G(X, Y) = 0 \), we have
\[
e_G(X, Y) \leq \delta_1 |X||Y| + \epsilon_1 t\ell |X| \leq \delta_1 |X||Y| + \epsilon_1 3|Y||X| \leq (\delta_1 + 3\epsilon_1)|X||Y| \leq \alpha(\gamma_1 - \alpha)(\gamma_2 - \alpha).
\]

Thus Definition 5.1(ii) holds. By (13) we have
\[
|X| \geq (\gamma_2 - \delta_1 - 2\epsilon_1)t\ell \geq (\gamma_2 - \delta_1 - 2\epsilon_1)(1 - \epsilon_1)n \geq (\gamma_2 - \delta_1 - 3\epsilon_1)n \geq (\gamma_2 - \alpha)n \quad \text{and}
\]
\[
|Y| \geq (\gamma_1 - \delta_1 - 2\epsilon_1)t\ell \geq (\gamma_1 - \delta_1 - 2\epsilon_1)(1 - \epsilon_1)n \geq (\gamma_1 - \delta_1 - 3\epsilon_1)n \geq (\gamma_1 - \alpha)n.
\]

Thus Definition 5.1(i) holds for some \( X' \subseteq X \), \( Y' \subseteq Y \) and \((X', Y')\) is an \( \alpha \)-splitting of \( G \).

Since \( G \) is connected, the claim follows immediately from (13) and Lemma 2.1. \( \square \)

Choose the notation so that \( \mathcal{P} = U_1 V_1 \ldots U_q V_q \). Add all clusters which are not in \( \mathcal{P} \) to the exceptional class \( U_0 \cup V_0 \). As \( \epsilon_1 \ll \delta_1 \), the exceptional class may now be much larger:
\[
|U_0| = |V_0| \leq 3\delta_1 n.
\]

Our next task is to reassign the vertices from the exceptional class to \( \mathcal{P} \). Since we will need to do this twice, we state the procedure in general terms. Let \( \{X_0, X_1, \ldots, X_q\} \cup \{Y_0, Y_1, \ldots, Y_q\} \) be the current partition, where \( \bigcup_{i=0}^q X_i = U \) and \( \bigcup_{i=0}^q Y_i = V \). Suppose that \((X_i, Y_i)\) and \((X_{i+1}, Y_i)\) are \( \epsilon' \)-regular pairs of density at least \( \delta' \). Recall that \((1 - \epsilon_1)\frac{n}{\ell} \leq \ell \leq \frac{n}{\sigma^2}\) was the common size of the non-exceptional clusters in the initial \( \epsilon_1 \)-regular partition. The procedure takes two parameters \( \sigma \) and \( \tau \) where \( \sigma^2 n \) is an upper bound on the size of the exceptional sets and \( 2\tau \ell \) is a minimum degree condition which a vertex must meet in order to be reassigned to a cluster. We arbitrarily group the vertices from \( X_0 \cup Y_0 \) into pairs \((u, v)\) and distribute them one pair at a time. In addition to reassigning vertices from \( X_0 \cup Y_0 \) we may move a vertex from one cluster to another. This process will be completed after \( s := |X_0| = |Y_0| \leq \sigma^2 n \) steps.

We use the following notation. For a cluster \( Z \) let \( Z^r \) denote \( Z \) after the \( r \)-th step of the reassignment. So \( Z = Z^0 \). Let \( O(Z^r) := Z^0 \cap Z^r \) denote the original vertices of \( Z^0 \) that remain after the \( r \)-th step, \( T(Z^r) := Z^r \setminus Z^0 \) denote the vertices that have been moved to \( Z \) during the first \( r \) steps, and \( F(Z^r) := Z^0 \setminus Z^r \) denote the vertices that have been moved from \( Z \) during the first \( r \) steps. We say that a cluster \( Z^r \) is full when \( |T(Z^r)| = \sigma \ell \).

**Procedure: Reassign**

For \( r = 1, \ldots, s \) reassign the \( r \)-th pair \((u, v)\) as follows:
we have conditions are satisfied: 

\( \epsilon \) density at least \( V \)

Lemma 8.3 (iv)

(ii) \( (i) \)

Reassign \( u \) to \( U_j^{r-1} \), \( v \) to \( V_i^{r-1} \), and if \( i \neq j \) then pick \( u' \in O(U_j^{r-1}) \) with \( \deg(u', V_i^0) \geq 2 \tau \ell \) and reassign \( u' \) to \( U_j^{r-1} \).

Lemma 8.3 (Reassigning Lemma). Suppose \( \{X_0, X_1, \ldots, X_q\} \cup \{Y_0, Y_1, \ldots, Y_q\} \) is a partition of \( V(G) \) in which the pairs \( (X_i, Y_i) \) and \( (X_{j+1}, Y_j) \) for \( i \in [q] \) and \( j \in [q-1] \), are \( \epsilon' \)-regular with density at least \( \delta' \), where \( 2\epsilon' \leq \delta' \), \( (1-d_0) \ell \leq |X_i|, |Y_i| \leq \ell \) and \( s = |X_0| = |Y_0| \leq \sigma^2 n \). If \( \epsilon_1 \leq \epsilon' \leq \sigma \leq \frac{1}{4} \tau \leq \frac{1}{4}d_0 \), then REASSIGN distributes all vertices from \( X_0 \cup Y_0 \) so that the following conditions are satisfied:

(i) If \( u \in T(X_i^\ell) \) then \( \deg(u, O(Y_i^{s})) \geq \tau \ell \) and if \( v \in T(Y_i^\ell) \) then \( \deg(v, O(X_i^s)) \geq \tau \ell \);

(ii) \( |X_i^\ell| - |Y_i^s| = |X_i^0| - |Y_i^0| \);

(iii) \( |T(X_i^\ell)|, |T(Y_i^s)| \leq \sigma \ell \) and \( |F(X_i^s)|, |F(Y_i^s)| \leq \sigma \ell \);

(iv) the pairs \( (O(X_i^s), O(Y_i^s)) \) and \( (O(X_{j+1}^s), O(Y_j^s)) \) are \( 2\epsilon' \)-regular with density at least \( \frac{1}{2} \delta' \).

**Proof.** Suppose that \( r \) pairs have been distributed and consider the \( (r+1) \)-th pair \((u,v)\). Let \( N'(u) = \{i : \deg(u, Y_i^0) \geq 2 \tau \ell \} \) and \( N'(v) = \{i : \deg(v, X_i^0) \geq 2 \tau \ell \} \).

Since 

\[ \gamma_2 n \leq \deg(v) \leq |N'(v)| \ell + 2 \tau \ell t + \sigma^2 n \leq |N'(v)| \frac{n}{t} + 2 \tau n + \sigma^2 n, \]

we have

\[ |N'(v)| \geq (\gamma_2 - 2 \tau - \sigma^2)t \geq (\gamma_2 - 3 \tau)t. \]
In the same way we obtain

$$|N'(u)| \geq (\gamma_1 - 3\tau)t.$$  

Now let

$$X = \bigcup_{i \in N'(u)} X_i^0 \subseteq U \text{ and } Y = \bigcup_{i \in N'(v)} Y_i^0 \subseteq V.$$  

Then we have

$$|Y| \geq |N'(v)|(1 - d_0)(1 - \epsilon_1)n \geq (\gamma_2 - 3\tau)(1 - d_0)(1 - \epsilon_1)n \geq (\gamma_2 - 5d_0)n \geq (\gamma_2 - \alpha)n.$$  

Similarly

$$|X| \geq (\gamma_1 - \alpha)n.$$  

Consequently, as the graph is not $\alpha$-splittable, we have

$$e(X, Y) > \alpha|X||Y| \geq \alpha(\gamma_1 - \alpha)(\gamma_2 - \alpha)n^2 \geq \alpha\gamma_1\gamma_2n^2/2. \quad (14)$$

Suppose that we are unable to distribute the pair $(u, v)$. We will derive a contradiction by counting edges incident with full clusters and edges in pairs $(U_i^\tau, V_j^\tau)$ with $e(U_i^\tau, V_j^\tau) < 3\tau\ell^2$. At most $s - 1 \leq \sigma^2n$ pairs of exceptional vertices have been distributed, and each time a pair is distributed there are at most two indices $i$ such that $|T(X_i^\tau)|$ or $|T(Y_i^\tau)|$ increases. Upon distribution, $|T(X_i^\tau)|$ or $|T(Y_i^\tau)|$ can increase by at most one. Thus there are at most

$$2\sigma^2n \over \sigma \ell = 2\sigma n \over \ell$$

pairs $(U_i, V_i)$ such that either $U_i$ or $V_i$ is full. The total number of edges of $G$ which are incident with vertices in these clusters is at most

$$4\sigma n \ell n = 4\sigma n^2.$$  

There are at most $3\tau n^2$ edges of $G$ in pairs $(X_i^0, Y_j^0)$ with $e(X_i^0, Y_j^0) < 3\tau\ell^2$. Then, since

$$(3\tau + 4\sigma)n^2 \leq 4\tau n^2 \leq \alpha\gamma_1\gamma_2n^2/2 < e(X, Y)$$

contradicts (14), there must exist $i \in N'(v)$ and $j \in N'(u)$ such that none of $X_i^\tau, Y_i^\tau, X_j^\tau, Y_j^\tau$ is full and $e(X_i^0, Y_j^0) \geq 3\tau\ell^2$. Then since $e(O(X_i^\tau), Y_j^0) \geq (3\tau - \sigma)\ell^2$ there is $u' \in O(X_i^\tau)$ with $
 degrade(u', Y_j^0) \geq 2\tau$. Thus the procedure distributes $(u, v)$.

Conditions (ii) and (iii) hold by design: for (iii) note that a vertex is only reassigned from a cluster if another vertex is reassigned to that cluster. Condition (iv) follows immediately from Lemma 7.4. Finally, condition (i) is satisfied since for every $u \in T(U_i^\tau)$ and $v \in T(V_i^\tau)$ we have

$$\deg(u, O(V_i^\tau)) \geq (2\tau - \sigma)\ell \geq \tau\ell \quad \text{and} \quad \deg(v, O(U_i^\tau)) \geq (2\tau - \sigma)\ell \geq \tau\ell.$$  

□
such that some of its neighbors. So now, in a similar way, we choose \( u \). So by Proposition 7.3 we can find vertices \( v \) all but at most \( \epsilon_2 \) of \( u \). Then \( O(U_i^1) = O(X_i^0) \), etc. By Lemma 8.3, each \( O(U_i^1), O(V_i^1) \) is \( \epsilon_2 \)-regular with density at least \( \epsilon_2 \) and \( \ell \geq |O(U_i^1)| = |O(V_i^1)| \geq (1-\epsilon_2)\ell \).

While \( (U_i^1, V_i^1) \) may not be \( \epsilon_2 \)-regular, the exceptional parts \( T(U_i^1) \) and \( T(V_i^1) \) satisfy:

\[
\forall u \in T(U_i^1), \forall v \in T(V_i^1), \deg(u, O(V_i^1)), \deg(v, O(U_i^1)) \geq d_0 \ell > \sqrt{3\delta_1} \ell \geq |T(U_i^1)|, |T(V_i^1)|.
\]

Our next goal is to find a small ladder in each pair \((U_i, V_i)\) which will contain all of the exceptional vertices \( T(U_i^1) \) and \( T(V_i^1) \). Precisely, we will prove the following.

**Claim 8.4.** For each \( i \in [r] \) there exists a ladder \( L_i \subseteq U_i^1 \cup V_i^1 \) such that:

(i) \( T(U_i^1) \cup T(V_i^1) \subseteq V(L_i) \).

(ii) \( |V(L_i)| \leq 16\sqrt{3\delta_1} \ell \).

(iii) Each \( w \in \text{ext}(L_i) \) satisfies \( \deg(w, (O(V_i^1) \cup O(U_i^1)) \setminus L_i) \geq \frac{1}{2} \epsilon_2 \ell \).

**Proof.** Let \( w_1, w_2, \ldots, w_s \) be an ordering of \( T(U_i^1) \cup T(V_i^1) \). Then \( s \leq 2\sqrt{3\delta_1} \ell \leq \frac{1}{12} d_0 \ell \). Suppose that we have constructed a ladder \( L \subseteq U_i^1 \cup V_i^1 \) on \( 8r \) vertices \((1 \leq r < s)\) that contains exactly the first \( r \) vertices of \( T(U_i^1) \cup T(V_i^1) \), satisfies (iii), and has first rung \( u'v' \) and last rung \( u''v'' \). Without loss of generality, assume that \( w_{r+1} \in T(U_i^1) \).

We will first show how to extend \( L \) to \( L' \) by attaching a 3-ladder \( aba'b'w_{r+1}v \), with \( a, a' \in O(U_i^1) \setminus L \) and \( b, b', v \in O(V_i^1) \setminus L \), to the end of \( L \) so that \( w_{r+1} \) and \( v \) satisfy (iii). By Lemma 7.2, all but at most \( \epsilon_2 \ell \) vertices \( v \in O(V_i^1) \) satisfy \( \deg(v, O(V_i^1) \setminus V(L)) \geq \frac{1}{2} \delta_2 \ell + 4 \). Choose such a vertex \( v \in N(w_{r+1}) \setminus V(L) \). Each \( x \in \{u'', v'', w_{r+1}, v\} \) has at least \( \frac{1}{2} \delta_2 \ell \) neighbors in \((O(V_i^1) \cup O(U_i^1)) \setminus L \).

So by Proposition 7.3 we can find vertices \( a, b, a', b' \in (O(V_i^1) \cup O(U_i^1)) \setminus L \) such that \( a \sim v'', b \sim u'', a' \sim v, b' \sim w_{r+1} \) and \( G[a, b, a', b'] = C_4 \), which completes the extension.

In extending \( L \) to \( L' \) we may have violated condition (iii) for the first rung \( u'v' \) by using up some of its neighbors. So now, in a similar way, we choose \( a'' \in O(U_i^1) \setminus L' \) and \( b'' \in O(V_i^1) \setminus L' \), such that \( u' \sim b'' \sim a'' \sim v' \) and \( \deg(a'', O(V_i^1) \setminus L') \), \( \deg(b'', O(U_i^1) \setminus L') \geq \frac{1}{2} \delta_2 \ell + 1 \). We then add \( a''b'' \) to \( L' \) as a first rung to obtain \( L'' \) satisfying (iii). Continuing in this fashion we obtain the desired ladder \( L' \) satisfying (i-iii).
Finally, let $(T, \delta)$. So by Lemma 7.5, since $\deg(w, U^2 \cup V^2) \leq (\delta - \epsilon_3)|V^2|$ is contained in $Q_i \cup R_i$. This is possible by Lemma 7.2.

Move the vertices in $Q_i \cup R_i$ to new exceptional sets to obtain the partition

$$U^3_0 := \bigcup_{i=1}^q Q_i, \quad V^3_0 := \bigcup_{i=1}^q R_i, \quad U^3_i := U^2_i \setminus Q_i, \quad \text{and} \quad V^3_i := V^2_i \setminus R_i.$$ 

Then $|U^3_0| = |V^3_0| \leq \epsilon_3 n$. By Lemma 7.4 the pairs $(U^3_i, V^3_i)$ are $(\epsilon_4, \delta_i)$-super-regular for $i \in [q]$. The pairs $(U^3_{j+1}, V^3_{j+1})$ may not be super-regular, but they are $(\epsilon_4, \delta)$-regular with density at least $\delta_4$.

Applying Lemma 8.3 to the partition $(U^3_0, U^3_1, \ldots, U^3_q) \cup (V^3_0, V^3_1, \ldots, V^3_q)$ with $\sigma = \sqrt{\epsilon_3}$ and $\tau = \delta_4$, we get a new partition $(U^3_0, U^3_1, \ldots, U^3_q) \cup (V^3_1, \ldots, V^3_q)$. Note that the pairs $(O(U^3_i), O(V^3_i))$ are $(\frac{1}{2} \epsilon_5, 2 \delta)$-super-regular and thus

$$(1 - d_0) \ell \leq (1 - 9 \sqrt{3 \delta_1} - \epsilon_3 - \sqrt{\epsilon_3}) \ell \leq |O(U^3_i)|, |O(V^3_i)| \leq \ell \quad \text{and}$$ 

$$|T(U^3_i)|, |T(V^3_i)| \leq \sqrt{\epsilon_3} \ell \leq \frac{1}{2} \sqrt{\epsilon_3} \ell \leq \sqrt{\epsilon_3} |O(U^3_i)|, \sqrt{\epsilon_3} |O(V^3_i)|.$$ 

So by Lemma 7.5, since $\deg(w, O(V^3_i)) \geq \delta_4|O(V^3_i)|$ and $\deg(w', O(U^3_i)) \geq \delta_4|O(U^3_i)|$, for all $w' \in T(U^3_i)$ and $w' \in T(V^3_i)$, the pairs $(U^3_i, V^3_i)$ are $(\epsilon, \delta)$-super-regular (with room to spare). Similarly, each pair $(U^3_{j+1}, V^3_{j+1})$ is $\epsilon$-regular with density at least $\delta$. Also $|U^3_i| = |V^3_i|$, except that $|V^3_i| = |U^3_i| + 1, |U^3_q| = |V^3_q| + 1$.

Using Lemma 7.2, for $i \in [q - 1]$, choose $v_i \in V^4_i$ such that $|A_{i+1}| \geq \frac{1}{2} \delta \ell$, where $A_{i+1} := U^4_{i+1} \setminus N(v_i)$. Similarly, choose $u_{i+1} \in A_{i+1}$ such that $|D_i| \geq \frac{1}{2} \delta \ell$, where $D_i := V^4_i \setminus N(u_{i+1})$. Set $P := \{v_i, u_{i+1} : i \in [q - 1]\}, U^5_i := U^4_i \setminus P$, and $V^5_i := V^4_i \setminus P$. Then (using the spared room) $(U^5_i, V^5_i)$ is still an $(\epsilon, \delta)$-super-regular pair. Now set $B_{i+1} := V^5_i \setminus N(u_{i+1})$ and $C_i := U^5_i \setminus N(v_i)$. Let $x_i, y_i$ be the first rung of $L'$ and let $w_i, z_i$ be the last rung of $L'$, where $x_i, w_i \in U$ and $y_i, z_i \in V$. Finally let $X_i := U^5_i \cap N(y_i), Y_i := V^5_i \cap N(x_i), W_i := U^5_i \cap N(z_i)$, and $Z_i := V^5_i \cap N(w_i)$. Note that each of $X_i, Y_i, W_i$, and $Z_i$ has size at least $\frac{1}{2} \delta \ell = \frac{1}{2} \ell$.

We now apply Lemma 7.9 to each pair $(U^5_i, V^5_i)$ to find a spanning ladder $M'$ whose first rung is contained in $A_i \times B_i, \epsilon_2$ whose second rung is contained in $X_i \times Y_i, \epsilon_2$ whose third rung is contained in $W_i \times Z_i$, and whose last rung is contained in $C_i \times D_i$. This is possible since $\eta \ell \geq 4$. Clearly we can insert $L'$ between the second and third rungs of $M'$ to obtain a ladder $L'$ spanning $U^4_i \cup V^4_i$. Finally, $L'^4v_1u_2L'^2 \ldots v_{r-1}u_rL'^r$ is a spanning ladder of $G$.

\begin{proof}{9} Proof of Amar’s Conjecture

Theorem 1.8 follows immediately from Lemmas 4.1, 6.1, 8.1 with $N_0(k) = \max\{N_1(k), N_2(k)\}$.

Now we prove Theorem 1.9.

\end{proof}{9}

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Let \( N_0(1) \) be the value given when \( k = 1 \) in Theorem 1.8 and set \( C := N_0(1) \). Suppose \( G \) is a balanced \( U, V \)-bigraph on \( 2n \) vertices with \( \delta_U + \delta_V \geq n + C \). We may assume without loss of generality that \( \delta_U = \delta(G) =: \delta \). We may assume \( \delta < \frac{n}{29} + 1 \), otherwise we would have a spanning ladder by Theorem 1.8 since the choice of \( C \) implies that \( n \geq N_0(1) \).

Let \( S = \{ x \in U : \deg(x) \leq \frac{9n}{10} \} \) and \( S' \subseteq S \) be a maximal subset such that \( |N(S')| < 3|S'| \). Let \( \bar{s} := |S| - |S'| \), then \( G[(S \setminus S') \cup (V \setminus N(S'))] \) contains a set of \( \bar{s} \) disjoint claws \( M = \{ a_r b_r c_r d_r : r \in [\bar{s}], a_r \in S \setminus S', b_r, c_r, d_r \in V \setminus N(S') \} \). We have the following bound on the cardinality of \( S \),

\[
(n - \delta + C)n \leq |E(G)| \leq \frac{9n}{10}|S| + n(n - |S|) \leq 10|S| - 10C.
\]

(15)

Note that for all \( v_1, v_2 \in V \cap V(M) \) we have

\[
|(N(v_1) \cap N(v_2)) \cap (U \setminus S)| \geq 2(n - \delta + C) - n - |S| > \frac{47}{50}n \geq 2\bar{s}.
\]

(16)

Thus by (16) there exists a set of 3-ladders

\[
\Lambda(M) = \{ x, a_r y_r b_r c_r d_r : r \in [\bar{s}], a_r, b_r, c_r, d_r \in M, x_r, y_r \in U \setminus S \}.
\]

Note that \( \text{ext}(L) \subseteq V(G) \setminus S \) for all \( L \in \Lambda(M) \). Let \( R = \bigcup_{L \in \Lambda(M)} V(L) \). For all \( v' \in V \setminus N(S') \), we have \( \deg(v') \geq n - \delta + C \), thus

\[
|S'| \leq \delta - C.
\]

(17)

Now we show that \( G \) contains a ladder that spans \( S' \). Let \( T = \{ x \in U : \deg(x) < n - 29\delta \} \). Then

\[
(n - \delta + C)n \leq |E(G)| < (n - 29\delta)|T| + n(n - |T|) \leq \frac{n}{29}.
\]

Let \( X' \) be any \((30\delta - |S'|)\)-subset of \( U \setminus (R \cup S \cup T) \) and \( U' = S' \cup X' \). Similarly, let \( Y' \) be any \((30\delta - |N(S')|)\)-subset of \( V \setminus (N(S') \cup V(M)) \) and \( V' = N(S') \cup Y' \). Let \( H := G[U' \cup V'] \). Then every vertex in \( X' \) is non adjacent to at most \( 29\delta \) vertices of \( V \) and so \( \delta_{V'} := \delta_{V'}(H) \geq \delta \). Similarly, \( \delta_{V'} := \delta_{V'}(H) \geq 29\delta + C \). Let \( m = 30\delta \) and note that \( \delta_{V'} + \delta_{V'} \geq m + C, \delta(H) \geq \frac{n}{30} \).
and by the choice of $C$, $m \geq N_0(1)$. Thus $H$ contains a spanning ladder $L = u_1v_1 \ldots u_{30\delta}v_{30\delta}$ by Lemmas 6.1 and 8.1. Since $|N(S')| < 3|S'|$ we have $|S' \cup N(S')| < 4\delta$ by (17). Thus there exists rungs $u_iv_i, u_{i+1}v_{i+1} \in E(L)$ with $2 \leq i \leq 30\delta - 2$ such that $u_i, v_i, u_{i+1}, v_{i+1} \in V(H) \setminus (S' \cup N(S'))$. Let $L_1 = u_1v_1 \ldots u_iv_i$ and $L_2 = u_{i+1}v_{i+1} \ldots u_{30\delta}v_{30\delta}$. We will specify $L_1$ as the initial ladder and $L_2$ as the terminal ladder. Let $\Lambda := \Lambda(M) \cup \{L_1, L_2\}$ and let $I = I(\Lambda) = \bigcup_{L \in \Lambda} \mathring{L}$. Set $q' := 0$, $s' := \bar{s} + 2 = |\Lambda|$ and $t' := 30\delta + 3\bar{s}$. Note that for all $z \in V(G) \setminus I$ we have,

$$\deg(z) \geq \frac{9n}{10} \geq \frac{3n + 100\delta}{4} + 1 \geq \frac{3n + 3s' + t' + 4q'}{4} + 1.$$ 

So we may apply Lemma 2.3 to $G$ to obtain a spanning ladder which starts with the first rung of $L_1$ and ends with the last rung of $L_2$.

Finally, we prove Theorem 1.10.

Proof. Let $C$ be the constant from Theorem 1.9, let $N_0(1) < N_0(2) < \cdots < N_0(C - 1)$ be the values given by Theorem 1.8, and let $N_0 = N_0(C - 1)$. Let $G$ be a balanced $U, V$-bigraph on $2n$ vertices with $n \geq N_0$ which satisfies $\delta_U + \delta_V \geq n + \operatorname{comp}(H)$. By Theorem 1.8 and Theorem 1.9, we have $H \subseteq G$.

References


