Consider the system of ODEs:
\[
\begin{align*}
\frac{dS}{dt} &= \Lambda - \beta S(t) \frac{I(t)}{N(t)} - \mu S + \gamma I \\
\frac{dI}{dt} &= \beta S(t) \frac{I(t)}{N(t)} - (\mu + \gamma)I
\end{align*}
\]
where \( \Lambda \) denotes the total recruitment rate, assumed constant.

1. Look at \( \frac{dN}{dt} = \frac{d(S+I)}{dt} \) and solve the resulting differential equation for \( N \), obtaining \( N(t) = K - (K - N(0))e^{-\mu t} \) where \( K = \frac{\Lambda}{\mu} \). This shows that the system is equivalent to the solution of the single \textit{nonautonomous} differential equation
\[
\frac{dI}{dt} = \beta(N(t) - I) \frac{I}{N(t)} - (\mu + \gamma)I.
\]

2. Show that \( N(t) \to K \) as \( t \to \infty \).

3. Choose \( K = 1000, \frac{1}{\mu} = 10 \) years, and two initial population sizes \( N(0) = 1200 \) and \( N(0) = 700 \). Also choose parameters so that \( R_0 = \frac{\beta}{\mu+\gamma} > 1 \). Use the numerical methods: (i) Euler (ii) midpoint and (iii) fourth order Adams Bashforth to obtain the solution \( I(t) \) up to time \( T = 50 \). You can use Euler with second order Adams Bashforth twice to start up the scheme.

Second order AB is:
\[
u^{n+1} = u^n + \frac{\Delta t}{2}(3f^n - f^{n-1})
\]

Answer the following questions:

(a) Compute the \textit{ode45} solution at time \( T = 20 \) with accuracy up to at least six digits. Consider this your exact solution, \( I_{exact}(t) \). Draw a plot of the corresponding \( I_{exact}(t) \).

(b) Choose eight different time steps \( \Delta t \) (make sure it converges) for each of the three schemes. Make a table listing \( |I_{exact}(20) - I(20)| \) for each scheme and for each \( \Delta t \).

(c) Draw a plot on log-log scale from the table you have generated as a function of number of evaluations of \( f^n \).

(d) What are the slopes of the lines in the graph above? Explain.

(e) Draw graphs of the “best” numerical solutions up until time \( T = 50 \).

\textbf{Bibliography}

1. Trefethen, 1994