MAT 444, Intermediate Abstract Algebra

Spring 2009

Professor Kadell

Final and Qualifying Examination, Friday May 8, 2009

Write your answers on separate paper.

Make sure your name is on every page of your answers.

I wish this examination to be graded as a Qualifying Examination in Abstract Algebra:

Yes

No

______________________________
Signature
Groups

1. (a) [5 points] State the Sylow theorems. Be sure to be complete and precise in your statements.
   (b) [5 points] Show that there are at least two non–isomorphic groups of order 10.
   (c) [20 points] Show that there are at most two non–isomorphic groups of order 10.

2. (a) State the following definitions for a group $G$:
   (i) [2 points] The conjugacy class $C_x$ of an element $x \in G$.
   (ii) [2 points] The centralizer $Z(x)$ of an element $x \in G$.
   (iii) [2 points] The center $Z$ of the group $G$.
   (b) [10 points] Show that $|C_x| | Z(x) | = |G|$.
   (c) [10 points] Let $|G| = p^s$ where $p \in \mathbb{Z}$ is prime and $s \geq 1$ is a positive integer. Show that $|Z| > 1$.
   (d) [10 points] Let $|G| = p^2$ where $p \in \mathbb{Z}$ is prime. Show that $Z(x) = G$ for all $x \in G$. Explain why this shows that $G$ is Abelian.

3. Let
   \[ G = \mathbb{Z}_{16} \times \mathbb{Z}_{16}/\langle(1\overline{0}, 1\overline{2})\rangle, \]  
   (1)
   where the canonical homomorphism $\theta : \mathbb{Z} \to \mathbb{Z}_{16}$ is given by $m \in \mathbb{Z} \to \theta(m) = \overline{m} = \{m + 16a \mid a \in \mathbb{Z}\}$, and
   \[ \langle(1\overline{0}, 1\overline{2})\rangle = \{(10a, 12a) \mid a \in 0, 1, \ldots, 15\} \]

   (a) Determine each of the following:
   (i) [2 points] The order of $1\overline{0}$ in $\mathbb{Z}_{16}$.
   (ii) [2 points] The order of $1\overline{2}$ in $\mathbb{Z}_{16}$.
   (iii) [2 points] The order of $(1\overline{0}, 1\overline{2})$ in $\mathbb{Z}_{16} \times \mathbb{Z}_{16}$.
   (iv) [2 points] The cardinality $|G|$ of the quotient group $G$ defined by (1) above.
(b) [6 points] Let $\phi : H \to K$ be a group homomorphism. Prove that for all $h \in H$, the order of $\phi(h)$ in $K$ divides the order of $h$ in $H$.

(c) [4 points] Prove that $G$ is not cyclic.

(d) [5 points] Find a generator for a cyclic subgroup $H < G$ of order $|H| = 16$, and justify your answer.

(e) [3 points] Show that there is a subgroup $K < G$ of order $|K| = 2$ with $H \cap K = \{e\}$ where $e = ((10, 12))$ is the identity element of $G$.

(f) [6 points] Determine the isomorphism type of the group $G$, stating carefully any theorems that you use.

Rings

4. Let $R$ be a commutative ring with one which is nonzero ($1 \neq 0$) and has no zero divisors ($ab = 0 \implies a = 0$ or $b = 0$).

(a) State the following definitions:
   (i) [2 points] $R$ is a principal ideal domain.
   (ii) [2 points] $r \in R$ is an irreducible element of $R$.
   (iii) [2 points] $r \in R$ is a prime element of $R$.
   (iv) [2 points] $R$ is a unique factorization domain.

(b) [5 points] Let $I_1 \subseteq I_2 \subseteq \ldots$ be an ascending chain of ideals of $R$. Show that $\bigcup_{n \geq 1} I_n$ is an ideal of $R$.

(c) [5 points] Let $a, b \in R$, $a \neq 0$ and $b$ is not a unit of $R$. Show that $(ab) \subsetneq (a)$.

(d) [7 points] Let $R$ be a principal ideal domain. Show that an irreducible element of $R$ is prime.

(e) [7 points] Let $R$ be a principal ideal domain and assume that $r \in R$ is not zero and is not a unit of $R$. Show that $r$ can be written as the product of finitely many irreducible elements of $R$.

5. Let

$$I = (x^4 + x^2 + 1, x^3 - 1)$$

$$= \{(x^4 + x^2 + 1)a(x) + (x^3 - 1)b(x) \mid a(x), b(x) \in \mathbb{Q}[x]\}$$

be the ideal of $\mathbb{Q}[x]$ generated by $x^4 + x^2 + 1$ and $x^3 - 1$.  


(a) [7 points] Explain briefly why there exists $p(x) \in \mathbb{Q}[x]$ such that
\[ I = (p(x)) = \{ p(x)a(x) \mid a(x) \in \mathbb{Q}[x] \}. \]  
(2)

(b) [10 points] Find a generator $p(x) \in \mathbb{Q}[x]$ for $I$ as given by (2) above.

(c) [8 points] Let
\[ J = (x, 3) = \{ xa(x) + 3b(x) \mid a(x), b(x) \in \mathbb{Z}[x] \} \]
be the ideal of $\mathbb{Z}[x]$ generated by $x$ and 3. Show that $J$ is not a principal ideal of $\mathbb{Z}[x]$.

6. (a) [9 points] Let $p \in \mathbb{Z}$ be prime. Show that $\mathbb{Z}_p$ is a field.

(b) [6 points] State Eisenstein’s criterion for irreducibility of polynomials in $\mathbb{Z}[x]$.

(c) [10 points] Prove Eisenstein’s criterion, stating carefully any theorems that you use.