REACTION-DIFFUSION EQUATIONS ON $\mathbb{R}^n$

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1. Introduction

These notes were created for use in various applied math courses I have taught in recent years in which I have discussed all or parts of it. The common theme is the scalar partial differential equation of reaction-diffusion type:

$$u_t = \Delta u + f(u)$$

where $u_t$ denotes the partial derivative w.r.t. variable $t$ and $\Delta u = \sum_{i} u_{x_i x_i}$ is the Laplacian of $u$. My favorite $f$ are the logistic function $f(u) = ru(1-u)$ and the cubic $f(u) = -au(b-u)(1-u)$ where $a > 0$ and $0 < b < 1$ manifesting the Allee effect from population biology. The PDE describes the density of a population of randomly motile organisms where $f(u)$ describes birth and death rates at population density $u$.

For applications to mathematical biology, I suggest starting with the last two sections. We consider how fast a non-indigenous species may invade a new territory and how large should a wildlife refuge be in order to sustain a mobile species if crossing the boundary of the refuge is lethal for the organism. The mathematical prerequisites needed for these sections is basic phase plane theory of two-dimensional systems of ODEs.

For students of partial differential equations interested in whether these equations are well-posed and give rise to dynamical systems, one can start at the beginning. I focus on the solutions of the PDE defined and bounded on $\mathbb{R}^n \times [0, T]$ for $T > 0$. The contraction mapping principle is used to show that solutions exists, at least for small $T$, and that exponentially bounded solutions exist on $\mathbb{R}^n \times [0, \infty)$ if $f$ is Lipschitz continuous. DuHamel’s principle for the nonhomogeneous heat equation is used to write an integral equation for so-called mild solutions of the PDE. These solutions are shown to be classical solutions of the PDE. We search for an appropriate state space for the PDE such that the solutions represent continuous motions in the space and that solutions are continuous with respect to the initial data. Finally, we
show that the dynamics have the monotonicity property: bigger initial states give rise to bigger future states.

2. Existence of Mild Solutions

We consider whether the initial boundary value problem

\begin{equation}
\begin{aligned}
    u_t &= \Delta u + f(u), \quad x \in \mathbb{R}^n, t > 0, \\
    u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\end{equation}

where $u_0$ is bounded and continuous and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. This implies that $f$ satisfies:

1. $\forall R > 0, \exists K > 0, \forall u_1, u_2, |u_i| \leq R, i = 1, 2, \Rightarrow |f(u_1) - f(u_2)| \leq K|u_1 - u_2|.$
2. $\forall R > 0, \exists M > 0, |u| \leq R \Rightarrow |f(u)| \leq M.$

Given $T > 0$, we seek a bounded continuous solution $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$. The notation $\|u\|_\infty$ and $\|u_0\|_\infty$ will be used for the supremum norms.

If such a solution exists, define $h(x, t) = f(u(x, t))$ and observe that $u$ satisfies

$$u_t = \Delta u + h(x, t).$$

By Duhamel's principle [1, 4], using the fundamental solution $\Phi(x, t)$ of the heat equation, we have

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s)f(u(y, s))dyds.$$

We call a bounded and continuous function $u$ a mild solution of (1) if it satisfies (2).

Recall that $\Phi$ is nonnegative and

$$\int_{\mathbb{R}^n} \Phi(x, t)dx = 1, \quad t > 0.$$

It is easy to see that the integrals in (2) are well-defined if $u$ is bounded and continuous.

For a given bounded continuous function $u_0$ and $R > 2\|u_0\|_\infty$, let

$$Z^R_T = \{ u : \mathbb{R}^n \times [0, T] \to \mathbb{R} : u \text{ is continuous, } \|u\|_\infty \leq R \}$$

Then $Z^R_T$ is a complete metric space with metric $d(u, v) = \|u - v\|_\infty$.

Define the map $F$ on $Z^R_T$ by the right hand side of (2):

$$F(u)(x, t) = \begin{cases} 
\int_{\mathbb{R}^n} \Phi(x - y, t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s)f(u(y, s))dyds, & t > 0 \\
0 & t = 0
\end{cases}.$$
Let the hypotheses of Theorem 1 hold and let

\[ \int_{\mathbb{R}^n} \Phi(y, t)u_0(x-y)dy + \int_0^T \chi_{[0,t)}(s) \int_{\mathbb{R}^n} \Phi(y, s)f(u(x-y, t-s))dyds, \quad t > 0 \]

Now it is easy to see that if \( (x_k, t_k) \to (x_0, t_0) \) with \( T \geq t_0 > 0 \), then \( F(u)(x_k, t_k) \to F(u)(x_0, t_0) \) by applying the Lebesgue dominated convergence theorem using that \( u_0 \) is bounded, \( f(u) \) is bounded, and \( \Phi(\bullet, t) \) is integrable. Continuity at \( t = 0 \) is proved just as in the case \( f = 0 \). Indeed, the second integral in the definition of \( F \), denoted here by \( I(x, t) \), satisfies \( |I(x, t)| \leq M t \) so it converges uniformly to zero.

Notice that we do not require that \( u \in Z^R_t \) satisfies \( u(x, 0) = u_0(x), x \in \mathbb{R}^n \) because any fixed point of \( F \) will satisfy this restriction.

Using (3), we find that if \( u \in Z^R_t \), then

\[ |F(u)(x, t)| \leq ||u_0||_\infty + MT \leq R/2 + MT, \quad (x, t) \in \mathbb{R}^n \times [0, T], \]

hence \( |F(u)|_\infty \leq R/2 + MT \) so if \( MT \leq R/2 \), then \( F: Z^R_t \to Z^R_t \).

Now we check that \( F \) is a contraction mapping on \( Z^R_t \).

\[
|F(u)(x, t) - F(v)(x, t)| \leq \int_0^t \int \Phi(x-y, t-s)|f(u(y, s)) - f(v(y, s))|dyds
\]

\[
\leq K \int_0^t \int \Phi(x-y, t-s)|u(y, s) - v(y, s)|dyds
\]

\[
\leq KT\|u - v\|_\infty,
\]

so

\[ ||F(u) - F(v)||_\infty \leq KT\|u - v\|_\infty, \quad u, v \in Z^R_t. \]

We conclude that if \( ||u_0||_\infty \leq R/2, MT \leq R/2 \) and \( KT \leq 1/2 \), then \( F \) is a contraction mapping on the complete metric space \( Z^R_t \). Let \( T = \min\{||u_0||_\infty/M, 1/2K\} \). Then the Contraction Mapping Theorem (Banach’s Fixed Point Theorem) [1], we have:

**Theorem 1.** Let \( f \) be continuously differentiable. Then for every bounded continuous \( u_0 \), there exists \( T > 0 \) and a unique bounded continuous mild solution \( u: \mathbb{R}^n \times [0, T] \to \mathbb{R} \) of (1). \( T \) depends only on \( ||u_0|| \) and \( f \).

2.1. **A mild solution is a classical solution.** We aim to show that the mild solution of Theorem 1 is a classical solution.

**Theorem 2.** Let the hypotheses of Theorem 1 hold and let \( u: \mathbb{R}^n \times [0, T] \to \mathbb{R} \) be the mild solution guaranteed to exist by Theorem 1. Then \( u \) is twice continuously differentiable with respect to \( x \) and once continuously differentiable with respect to \( t \) in \( \mathbb{R}^n \times (0, T] \) and satisfies (1).
The proof is based on Theorem 5.2.11 in [4] which establishes that if \( h(x, t) \) and \( \nabla_x h(x, t) \) are bounded and continuous on \( \mathbb{R}^n \times (0, T] \) then the solution of the integral equation provided by Duhamel’s principle for the initial-boundary value problem

\[
v_t = \Delta v + h(x, t), \quad v(\cdot, 0) = u_0,
\]

has the regularity properties described in Theorem 2 and satisfies the PDE on the domain indicated.

We temporarily assume that \( u_0 = 0 \) and follow the argument in [4] that establishes the validity of DuHamel’s formula for (4). A preamble to the proof of Theorem 5.2.11 establishes that the first derivatives with respect to the \( x_i \) of the Duhamel solution \( v \) of (4) exist and are bounded on \( \mathbb{R}^n \times (0, T] \) even if \( h \) is merely continuous and bounded. These derivatives are given by formula (5.2.6) in [4]:

\[
v_{x_i}(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi_{x_i}(x - y, t - s) h(y, s) ds
\]

and the estimate

\[
|\Phi_{x_i}(x - y, t - s)| \leq (4\pi(t - s))^{-n/2} \left| y_i \right| + R \frac{e^{-\left| y \right|^2 + 2|y|s}}{4(t-s)^{n/2}}, \quad |x| \leq R,
\]

can be used to justify the differentiation under the integral sign since the right side is integrable and \( h \) is bounded. The continuity and boundedness of \( v_{x_i}(x, t) \) can be established by changing variables in the integral above such that \( (x - y, t - s) \) appears in the argument of \( h \) rather than \( \Phi_{x_i} \).

Let \( u \) be the mild solution given by Theorem 1 with general \( u_0 \) and let \( h(x, t) = f(u(x, t)) \). Then \( h \) is bounded and continuous on \( \mathbb{R}^n \times [0, T] \) so the argument in the proof of Theorem 5.2.11 shows that the \( u_{x_i} \) are bounded and continuous on this same domain and therefore \( h_{x_i} \) are bounded and continuous by the chain rule. Denote by \( v \) the solution of (4) with this \( h \). By Theorem 5.2.11 in [4], \( v = F(u) \) since \( F(u) \) precisely describes DuHamel’s formula. By Theorem 5.2.11, \( v \) has the regularity properties indicated in Theorem 2 and satisfies (4) in the classical sense. As \( u = F(u) \) and \( v = F(u) \), it follows that \( u = v \), proving Theorem 2.

2.2. A global existence theorem. Here we seek a fixed point of \( F \) but in a different space.

Define the family of norms

\[
\|u\|_r = \sup \{e^{-rt}|u(x, t)| : (x, t) \in \mathbb{R}^n \times [0, \infty)\}, \quad r > 0.
\]
These norms discount the future. Let
\[ B_r = \{ u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} : u \text{ is continuous, } \| u \|_r < \infty \}, \quad r > 0. \]
Observe that a continuous function \( u \) belongs to \( B_r \) if and only if there exists \( M \geq 0 \) such that
\[ |u(x, t)| \leq Me^{rt}, \quad x \in \mathbb{R}^n, t > 0, \]
and that if \( u \in B_r \) and \( T > 0 \), then
\[ (5) \quad \sup\{|u(x, t)| : x \in \mathbb{R}^n, 0 \leq t \leq T\} \leq e^{rT}\|u\|_r. \]
It is easy to see that \( B_r \) is a complete metric space and convergence of a sequence in \( B_r \) implies uniform convergence on each strip \( \mathbb{R}^n \times [0, T] \).

Now suppose that \( f \) is Lipschitz on \( \mathbb{R} \) with Lipschitz constant \( K \).

Let's verify that \( F \) maps \( B_r \) into itself. Let \( u \in B_r \) for some \( r > 0 \).

Then
\[
e^{-rt}|F(u)(x, t)| \leq e^{-rt} \int_{\mathbb{R}^n} \Phi(x - y, t) |u_0(y)| dy + e^{-rt} \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) |f(u(y, s))| dy ds
\]
\[ \leq \|u_0\|_\infty + e^{-rt}|f(0)| \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) e^{-rs} |u(y, s)| dy ds \]
\[ + K \int_0^t e^{-r(t-s)} \int_{\mathbb{R}^n} \Phi(x - y, t - s) e^{-rs} |u(y, s)| dy ds \]
\[ \leq \|u_0\|_\infty + |f(0)| e^{-rt} + K \|u\|_r, \int_0^t e^{-r(t-s)} ds \]
\[ \leq \|u_0\|_\infty + |f(0)| e^{-rt} + (K/r) \|u\|_r, \]
where we used that \( f(u) = f(0) + f(u) - f(0) \). Thus, \( \|F(u)\|_r \leq \|u_0\|_\infty + |f(0)| e^{-rt} + (K/r) \|u\|_r \) and, in particular, \( F(u) \in B_r \). Indeed, the continuity of \( F(u) \) is proved as in the previous section.

We show that \( F \) is a contraction on \( B_r \) with norm \( \|u\|_r \), if \( r \) is sufficiently large. Let \( u, v \in B_r \). Then
\[
e^{-rt}|F(u)(x, t) - F(v)(x, t)| \leq e^{-rt} \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) |f(u(y, s)) - f(v(y, s))| dy ds
\]
\[ \leq K \int_0^t e^{-r(t-s)} \int_{\mathbb{R}^n} \Phi(x - y, t - s) e^{-rs} |u(y, s) - v(y, s)| dy ds \]
\[ \leq K \|u - v\|_r \int_0^t e^{-r(t-s)} ds \]
\[ \leq (K/r) \|u - v\|_r. \]
Therefore, we have
\[ \|F(u) - F(v)\|_r \leq (K/r)\|u - v\|_r, \ u, v \in B_r. \]
If we choose \( r = 2K \), then \( F \) is a contraction on \( B_r \) and we get:

**Theorem 3.** Let \( f \) satisfy a Lipschitz condition with constant \( K \) on \( \mathbb{R} \). Then for each bounded and continuous initial data \( u_0 \), there exists a unique mild solution \( u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) in \( B_r \) for \( r = 2K \). In fact, \( \|u\|_r \leq 2\|u_0\| + \left| f(0) \right| e^{2Kt} / K e \), i.e.,

\[ |u(x, t)| \leq \left[ 2\|u_0\|_{\infty} + \frac{|f(0)|}{Ke} \right] e^{2Kt}, \ x \in \mathbb{R}^n, t > 0. \]


The final estimate of \( \|u\|_r \) follows by applying the same estimates used in the argument that \( F \) maps \( B_r \) into itself and uses \( r = 2K \).

A sharper estimate can be obtained by Gronwall’s inequality. Here, we assume that \( f(0) = 0 \) for simplicity.

\[ |u(x, t)| \leq \int_{\mathbb{R}^n} \Phi(x - y, t)|u_0(y)|dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s)|f(u(y, s))|dyds \]
\[ \leq \|u_0\|_{\infty} + K \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s)|u(y, s)|dyds \]
\[ \leq \|u_0\|_{\infty} + K \int_0^t \|u(\cdot, s)\|_{\infty}ds, \]

implying that

\[ \|u(\cdot, t)\|_{\infty} \leq \|u_0\|_{\infty} + K \int_0^t \|u(\cdot, s)\|_{\infty}ds. \]

We are using \( \|u(\cdot, s)\|_{\infty} \equiv \sup_{x \in \mathbb{R}^n} |u(x, s)| \). Gronwall’s inequality then gives

\[ \|u(\cdot, t)\|_{\infty} \leq \|u_0\|_{\infty} e^{Kt}, \ t \geq 0. \]

The case that \( f \) is linear, \( f(u) = ru \), can be solved explicitly. If \( u \) satisfies

\[ u_t = \Delta u + ru \]

multiply by integrating factor \( e^{-rt} \) and integrate to find that \( v = e^{-rt}u \) satisfies the heat equation. Therefore,

\[ u(x, t) = \int_{\mathbb{R}^n} e^{rt}\Phi(x - y, t)u_0(y)dy. \]

Note that, regardless of the sign of \( r \), \( u_0 \geq 0 \) implies that \( u(x, t) \geq 0 \) and \( u(x, t) > 0 \) if \( u_0(x_0) > 0 \) for some \( x_0 \).
3. Bigger initial data implies bigger solutions

We continue to assume that \( f \) satisfies a Lipschitz condition with constant \( K \). Rewrite (1) as

\[
u_t + Ku = \Delta u + f(u) + Ku.
\]

Let \( g(u) = f(u) + Ku \) and multiply both sides by integrating factor \( e^{Kt} \) to get that \( v = ue^{Kt} \) satisfies

\[v_t = \Delta v + e^{Kt}g(ve^{-Kt})\]

Then DuHamel’s principle implies that

\[
v(x, t) = \int_{\mathbb{R}^n} \Phi(x-y, t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)e^{Ks}g(v(y, s)e^{-Ks})dyds.
\]

Equivalently,

\[
u(x, t) = \int_{\mathbb{R}^n} e^{-Kt}\Phi(x-y, t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} e^{-K(t-s)}\Phi(x-y, t-s)g(u(y, s))dyds.
\]

The primary virtue of passing from \( f \) to \( g \) is that the latter is non-decreasing. For if \( u \le v \), then \( g(v) - g(u) = f(v) - f(u) + K(v - u) \ge -K(v - u) + K(v - u) = 0 \). Also, \( g \) is Lipschitz with constant \( 2K \).

Let \( \Psi_t(x) = \Psi(x, t) = e^{-Kt}\Phi(x, t) \) where the subscript \( t \) does not denote partial derivative. This notation is useful as we may write the first integral as convolution

\[
\Psi_t * u_0 = \int_{\mathbb{R}^n} \Psi(x-y, t)u_0(y)dy
\]

If we set \( u_t(x) = u(x, t) \), then the integral equation may be elegantly expressed as

\[
u_t = \Psi_t * u_0 + \int_0^t \Psi_{t-s} * g(u_s)ds.
\]

Global existence of a solution of (6) in \( B_r \) can be obtained in the same way as in Theorem 3. A straightforward estimate using that \( g \) has Lipschitz constant \( 2K \) gives

\[
\|F(u)\|_r \leq \|u_0\|_\infty + \frac{2K}{K + r}\|u\|_r
\]

and the choice \( r = 2K \) gives rise to a contraction mapping \( F : B_{2K} \to B_{2K} \).

Consider the initial value problem for the ODE \( y' = f(y), y(0) = y_0 \).
It is a direct consequence of uniqueness of solutions that if \( y_0 \leq z_0 \) and
if $y(t)$ is the solution with initial data $y_0$ and $z(t)$ is the solution with initial data $z_0$, then $y(t) \leq z(t)$ for $t \geq 0$. The same holds for the (1).

**Theorem 4.** Let $u_0$ and $v_0$ be bounded and continuous functions on $\mathbb{R}^n$ and denote by $u$ and $v$ the unique mild solutions in $B_{2K}$. If $u_0 \leq v_0$ on $\mathbb{R}^n$, then $u \leq v$ on on $\mathbb{R}^n \times [0, \infty)$.

**Proof.** We expand the notation for $F$ to include the initial data: $F(u) = F(u; u_0)$. By hypothesis, $u_0 \leq v_0$ which implies that $u = F(u; u_0) \leq F(u; v_0)$. Set $v^1 = F(u; v_0)$ so $v^1 \geq u$. As $g$ is nondecreasing, $v^2 \equiv F(v^1; v_0) \geq F(u; v_0) = v^1$. Define, $v^{k+1} = F(v^k; v_0)$ for $k \geq 2$. If $v^k \geq v^{k-1}$ for some $k \geq 2$, then $v^{k+1} = F(v^k; v_0) \geq F(v^{k-1}; v_0) = v^k$. Hence, the sequence $\{v^k\}_k$ is monotone with $v^k \geq u$ for all $k$. By the contraction mapping theorem, the sequence converges to $v$ and therefore $v \geq u$. □

Below, we apply Theorem 4 to the logistic equation.

**Corollary 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz with constant $K$ and $f(u) = u(1 - u)$ for $0 \leq u \leq 1$ and let $u_0$ satisfy $0 \leq u_0(x) \leq 1$ for all $x$. Then the unique mild solution $u$ of (1) satisfies $0 \leq u(x, t) \leq 1$ for all $x$ and $t \geq 0$.

**Proof.** This is an immediate consequence of Theorem 4, since both zero and one are equilibrium solutions: $F(0; 0) = 0$ and $F(1; 1) = 1$. Since $0 \leq u_0 \leq 1$, we have $0 \leq u \leq 1$. □

### 3.1. Stability of an equilibrium.

An equilibrium solution of (1) is a bounded function $v$ satisfying

\[ 0 = \Delta v + f(v) \tag{7} \]

We say that $v$ is stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all bounded continuous initial data $u_0$ satisfying $\|u_0 - v\|_{\infty} < \delta$, then $\|u_t - v\|_{\infty} < \epsilon$, $t > 0$. Otherwise, $v$ is unstable. It is asymptotically stable if it is stable and there exists $\eta > 0$ such that if $\|u_0 - v\|_{\infty} < \delta$, then $\lim_{t \to \infty} \|u_t - v\|_{\infty} = 0$.

We can use Theorem 4 together with solutions of the ODE $y' = f(y)$ to determine the stability of spatially constant equilibria $v$ (i.e. it satisfies $f(v) = 0$) since solutions of the ODE are also solutions of the PDE (1). Below is an example of this.

**Corollary 2.** Let $v$ be a spatially constant equilibrium of (1). If $v$ is stable, respectively unstable, respectively asymptotically stable for the ODE, then it has this property for the PDE (1).
Proof. If \( v \) is stable for the ODE, then for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |y(0) - v| < \delta \) implies \( |y(t) - v| < \epsilon \) for \( t > 0 \). Let \( y^\pm(t) \) be the solutions of the ODE with initial data \( y^\pm(0) = v \pm \delta/2 \). If \( \|u_0 - v\|_\infty < \delta/2 \), then it follows from Theorem 4 that \( y^-(t) < u(x, t) < y^+(t), \ t > 0 \) so

\[-\epsilon < y^-(t) - v < u(x, t) - v < y^+(t) - v < \epsilon, \ x \in \mathbb{R}^n, t > 0.\]

This proves that \( v \) stable for the PDE. Asymptotic stability of \( v \) for the PDE is proved similarly by comparison. Instability of \( v \) for the ODE implies instability of \( v \) for the PDE since solutions of the ODE can be viewed as solutions of the PDE.

\[
\text{Corollary 3. Let } f : \mathbb{R} \rightarrow \mathbb{R} \text{ be Lipschitz with constant } K \text{ and } f(u) = u(1 - u) \text{ for } -1 \leq u \leq 2. \ Then the equilibrium } 1 \text{ is asymptotically stable and the equilibrium } 0 \text{ is unstable.}
\]

4. Dynamical System

Let \( BC \) be the Banach space of bounded and continuous functions on \( \mathbb{R}^n \) and observe that our global solutions \( u \in B_{2K} \) satisfy \( u_t \equiv u(\bullet, t) \in BC \) for all \( t \geq 0 \) by (5). Our goal is to show that (6) generates a dynamical system on the state space \( BC \) via

\[ S(t)(u_0) = u_t, \ t \geq 0, \]

where \( S(t) : BC \rightarrow BC, \ t \geq 0, \) satisfies

\[ S(0) = I, \ S(t) \circ S(t_0) = S(t + t_0), \ t, t_0 \geq 0. \]

(8)

\[ u_{t+t_0} = S(t)u_{t_0} = u_t, \ t \geq 0, \]

\[
\begin{align*}
 u_{t+t_0} &= \Psi_{t+t_0} \ast u_0 + \int_0^{t+t_0} \Psi_{t+t_0-s} \ast g(u_s)ds \\
 &= \Psi_t \ast \Psi_{t_0} \ast u_0 + \int_0^{t_0} \Psi_t \ast \Psi_{t_0-s} \ast g(u_s)ds + \int_{t_0}^{t+t_0} \Psi_{t-t_0-s} \ast g(u_s)ds \\
 &= \Psi_t \ast \left( \Psi_{t_0} \ast u_0 + \int_0^{t_0} \Psi_{t_0-s} \ast g(u_s)ds \right) + \int_0^t \Psi_{t-\eta} \ast g(u_{t_0+\eta})d\eta \\
 &= \Psi_t \ast u_{t_0} + \int_0^t \Psi_{t-\eta} \ast g(u_{t_0+\eta})d\eta.
\end{align*}
\]

This implies that \( t \rightarrow u_{t+t_0} \) satisfies the same integral equation as the solution corresponding to initial data \( u_{t_0} \) and therefore, by uniqueness of solutions, \( S(t)(u_{t_0}) = S(t)S(t_0)u_0 = S(t + t_0)u_0 \).
What topology on $BC$ makes the maps $u \to S(t)u$ continuous for each $t > 0$? If $t \to t_0$, does $S(t)u_0 \to S(t_0)u_0$? Here we investigate continuity properties of the semiflow.

Consider the special case that $g = 0$ so the equation (1) is linear so we can write the solution explicitly

$$u_t(x) = \int_{\mathbb{R}^n} \Psi(x-y, t)u_0(y)dy$$

Then we can estimate that for $t > 0$

$$|u_t(x) - u_t(z)| \leq \|u_0\|_\infty \int_{\mathbb{R}^n} |\Psi(x-y, t) - \Psi(z-y, t)|dy$$

$$= \|u_0\|_\infty \int_{\mathbb{R}^n} |\Psi(\eta, t) - \Psi(\eta + w, t)|d\eta$$

$$= \|u_0\|_\infty \int_{\mathbb{R}^n} \left| \int_0^1 \nabla_x \Psi_t(\eta + sw) \cdot wds \right| d\eta$$

$$= \|u_0\|_\infty \int_{\mathbb{R}^n} \left| \int_0^1 \left(\frac{\eta + sw}{t}\right) \Psi_t(\eta + sw) \cdot wds \right| d\eta$$

$$\leq \|u_0\|_\infty \int_{\mathbb{R}^n} |\xi| \Psi_t(\xi) d\xi \left| \frac{x-z}{t} \right|$$

where $w = z - x$ and we used that $\nabla_x \Phi_t = -\frac{x}{t} \Phi_t$. It follows that $u_t : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz and hence uniformly continuous for each $t > 0$. But the bounded and continuous function $u_0$ may not be uniformly continuous. In that case, $u_t$ cannot converge uniformly to $u_0$ since the uniform limit of uniformly continuous functions is uniformly continuous. It follows that the map $t \to u_t \in BC$ is not continuous at $t = 0$ if $u_0$ is not uniformly continuous.

Therefore, we should consider our state space to be $BUC$, the space of bounded and uniformly continuous functions. It is a Banach space with the supremum norm.

The next result establishes that the initial value problem for (1) is well-posed in $BUC$: solutions depend continuously on the initial data over finite time intervals.

**Theorem 5.** If $u_0 \in BUC$ and $u_t = S(t)u_0$ is the solution of (6), then $t \to u_t$ is a continuous map from $[0, \infty)$ to $BUC$.

If $u_0, v_0 \in BUC$ and $u_t = S(t)u_0, \ v_t = S(t)v_0$ then

$$\|u_t - v_t\|_\infty \leq \|u_0 - v_0\|_\infty e^{2Kt}, \ t \geq 0.$$  

**Proof.** Consider the first assertion. By (8), it suffices to show that the map is continuous at $t = 0$. Let $M > 0$ be such that $\|u_t\|_\infty \leq M, \ 0 \leq}$
$t \leq 1$. Since $u_t - u_0 = \Psi_t * u_0 - u_0 + \int_0^t \Psi_{t-s} * g(u_s)ds$, we first show that the integral converges to zero uniformly in $x \in \mathbb{R}^n$:

$$\left| \int_0^t \Psi_{t-s} * g(u_s)ds \right| \leq K \int_0^t \Psi_{t-s} * |u_s|ds \leq KM \int_0^t \Psi_{t-s} * 1ds = KM \int_0^t e^{-K(t-s)}ds = M(1 - e^{-Kt})$$

Now, consider $\Psi_t * u_0 - u_0$. We have

$$|\langle \Psi_t * u_0 \rangle(x) - u_0(x)| \leq \int_{\mathbb{R}^n} \Psi_t(y)|u_0(x) - u_0(x - y)|dy = \int_{|y|<\delta} \Psi_t(y)|u_0(x) - u_0(x - y)|dy + \int_{|y|\geq\delta} \Psi_t(y)|u_0(x) - u_0(x - y)|dy \leq \int_{|y|<\delta} \Psi_t(y)|u_0(x) - u_0(x - y)|dy + 2M \int_{|y|\geq\delta} \Psi_t(y)dy$$

The final integral converges to zero as $t \to 0$ just as in the proof of Theorem 1, page 47 [1]. Since $u_0$ is uniformly continuous, given $\epsilon > 0$, there exists $\delta > 0$ such that $|u_0(x) - u_0(x - y)| < \epsilon$ if $|y| < \delta$ for all $x \in \mathbb{R}^n$. Now the proof proceeds as in [1]. This completes the proof of the first assertion.

As for the second assertion, we observe that:

$$|u_t(x) - v_t(x)| \leq |\Psi_t * (u_0 - v_0)| + \int_0^t |\Psi_{t-s} * (g(u_s) - g(v_s))|ds \leq e^{-Kt}\|u_0 - v_0\|_\infty + 2K \int_0^t \Psi_{t-s} * |u_s - v_s|ds \leq \|u_0 - v_0\|_\infty + 2K \int_0^t e^{-K(t-s)}\|u_s - v_s\|_\infty ds \leq \|u_0 - v_0\|_\infty + 2K \int_0^t \|u_s - v_s\|_\infty ds$$

Now apply Gronwall’s inequality, noting that $s \to \|u_s - v_s\|_\infty$ is continuous by the first part of the proof. \qed
5. Hölder spaces

If \( u : \mathbb{R}^n \to \mathbb{R} \) and \( 0 < \alpha \leq 1 \), define

\[
[u]_\alpha = \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} : x \neq y \right\}
\]

If finite, then obviously \( u \) is uniformly continuous on \( \mathbb{R}^n \) and

\[
|u(x) - u(y)| \leq [u]_\alpha |x - y|^{\alpha}, \quad x, y \in \mathbb{R}^n.
\]

For \( r > 0, 0 < \alpha \leq 1 \), and \( C > 0 \) let

\[
BH = \{ u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} : u \in B_r, \sup_{t \geq 0} e^{-rt}[u]_\alpha \leq C \}.
\]

\( BH \) is a complete metric space employing the norm \( \|u\|_r \). Indeed, \( BH \) is a closed subset of the Banach space \( B_r \). Recall that convergence in \( B_r \) implies uniform convergence on \( \mathbb{R}^n \times [0, T] \) for each \( T > 0 \).

Now, suppose that \( u_0 \in BC \) and \( [u_0]_\alpha \leq C \). We will show that our mapping \( F : B_r \to B_r \) maps \( BH \) into itself for \( r = 2K \). As we showed that \( F \) is a contraction mapping on \( B_{2K} \), this will show that the fixed point \( u \in BH \).

Let \( u \in BH \) with \( r = 2K \). Then

\[
|F(u_t)(x) - F(u_t)(z)| \leq \int_{\mathbb{R}^n} \Psi_t(y)|u_0(x - y) - u_0(z - y)|dy \\
+ \int_0^t \int_{\mathbb{R}^n} \Psi_{t-s}(y)|g(u_s(x - y)) - g(u_s(z - y))|dyds \\
= [u_0]_\alpha |x - z|^{\alpha} e^{-Kt} \\
+ 2K \int_0^t \int_{\mathbb{R}^n} \Psi_{t-s}(y)|u_s(x - y) - u_s(z - y)|dyds \\
\leq \left( Ce^{-Kt} + 2K \int_0^t [u_s]_\alpha e^{-K(t-s)} ds \right) |x - z|^{\alpha} \\
\leq e^{-Kt} \left( C + 2KC \int_0^t e^{3Ks} ds \right) |x - z|^{\alpha} \\
\leq e^{2Kt} C |x - z|^{\alpha}
\]

This yields that

\[
e^{-2Kt}[F(u_t)]_\alpha \leq C.
\]

6. Strong Parabolic Maximum Principle on \( \mathbb{R} \)

The following theorem can be found in [2], although with no proof.
**Theorem 6.** Let \( v, v_t, v_{xx} \) be continuous on \( \mathbb{R} \times [0, T] \), \( v \) bounded, and
\[
\begin{align*}
v_t - v_{xx} - c(x, t)v & \geq 0, \ (x, t) \in \mathbb{R} \times (0, T) \\
v(x, 0) & \geq 0, \ x \in \mathbb{R},
\end{align*}
\]
where \( c \) is bounded on \( \mathbb{R} \times [0, T] \). Then \( v(x, t) \geq 0 \).

Moreover, if \( v(x_0, t_0) = 0 \) for some \( (x_0, t_0) \) with \( t_0 > 0 \), then \( v(x, t) \equiv 0 \) for \( (x, t) \in \mathbb{R} \times [0, t_0] \).

**Proof.** \( u = e^{\mu t}v \) satisfies \( u_t = \mu u - e^{\mu t}v_t \) so
\[
\begin{align*}
u_t - u_{xx} - (c(x, t) + \mu)u & \leq 0, \ (x, t) \in (0, T) \\
u(x, 0) & \leq 0, \ x \in \mathbb{R},
\end{align*}
\]
Choose \( \mu < 0 \) so \( h \equiv c + \mu < 0 \). Then
\[
u_{xx} + hu - u_t \geq 0
\]
and it follows that \( u \) cannot have a positive local maximum at any point of \( \mathbb{R} \times (0, T) \).

Let \( w(x, t) = e^{\lambda t} \cosh x \). Then \( w_{xx} = w \) and \( w_t = \lambda w \) so
\[
w_{xx} + hw - w_t = (1 + h - \lambda)w \leq 0
\]
if \( \lambda > 0 \) is large enough. This trick is due to [5] Theorem 8.1.4.

Let \( m > 0 \) be such that \( |u(x, t)| \leq m \) in \( \mathbb{R} \times [0, T] \). Let \( R > 0 \) and define
\[
z = u - \frac{m}{\cosh R}w, \ (x, t) \in [-R, R] \times [0, T].
\]
Then
\[
z_{xx} + hz - z_t = u_{xx} + hu - u_t - \frac{m}{\cosh R}(w_{xx} + hw - w_t) \geq 0, \ (x, t) \in [-R, R] \times (0, T].
\]
Moreover, \( z(x, 0) \leq u(x, 0) \leq 0 \), \( -R < x < R \), and
\[
z(R, t) = u(R, t) - me^{\lambda t} \leq m(1 - e^{\lambda t}) \leq 0, \ 0 \leq t \leq T.
\]
Similarly, for \( z(-R, t) \), \( z \) cannot attain a positive maximum in \( (-R, R) \times (0, T) \) and it is non-positive on the parabolic boundary of \( [-R, R] \times [0, T] \). It follows that \( z(x, t) \leq 0, \ (x, t) \in [-R, R] \times (0, T] \). Equivalently,
\[
u(x, t) \leq \frac{m \cosh x}{\cosh R} e^{\lambda t}, \ (x, t) \in [-R, R] \times [0, T].
\]
As this inequality is valid for all large \( R \), we conclude that \( u(x, t) \leq 0 \) and therefore, \( v(x, t) \geq 0 \) as desired.

The final assertion follows by the above construction and by application of Theorem 4 in Protter & Weinberger, p 172, to the equation for \( u \).

\[\square\]
7. How a non-indigenous species invades a new territory

In invading species entering a new habitat might be modeled by the diffusive logistic equation.

\[ u_t = u_{xx} + ru(1 - u), \quad x \in \mathbb{R}, \ t > 0. \tag{10} \]

where \( r > 0 \).

Look for traveling wave solution \( u(x, t) = v(x - ct) \), with \( c > 0 \), satisfying

\[ v(-\infty) = 1, \ v(+\infty) = 0, \ v'(s) < 0, v(s) > 0, \tag{11} \]

representing an invading species.

Inserting the traveling wave ansatz into (10), we see that it satisfies

\[ v'' + cv' + rv(1 - v) = 0 \tag{12} \]

or, in system form,

\[
\begin{align*}
v' &= w \\
w' &= -cw - rv(1 - v)
\end{align*}
\]

Our solution should connect the equilibria \( E_1 = (1, 0) \) and \( E_0 = (0, 0) \).

The nullclines \( v' = 0 = w \) and \( w' = 0 \), equivalently, \( w = -\frac{r}{c}v(1 - v) \), will play a crucial role. From (11), we see that we are seeking a trajectory with \( w < 0 \) and \( 0 \leq v \leq 1 \) that leaves \( E_1 \) and arrives at \( E_0 \). This trajectory must be part of the unstable manifold of \( E_1 \).

The Figure below, using MATLAB pplane8, shows the nullclines in orange and purple, equilibria in red, and stable and unstable manifolds of \( E_1 \) in green.
The Jacobian matrix at $E_1$:

$$
\begin{pmatrix}
0 & 1 \\
-1 & -c
\end{pmatrix}
$$

with trace $-c$ and determinant $-r < 0$ so it is a saddle point.

The Jacobian matrix at $E_0$:

$$
\begin{pmatrix}
0 & 1 \\
-r & -c
\end{pmatrix}
$$

with trace $-c$ and determinant $r > 0$ so it is asymptotically stable.

We must have the eigenvalues of $E_0$ be real for if they are complex then $E_0$ is a spiral point so all solutions must spiral around it as they approach it, guaranteeing that $v$ is negative for infinitely many values of its argument.

Therefore, we must require that

$$(13) \quad c^2 \geq 2r$$

so the discriminant is positive.

Now, the "left part" of the unstable manifold of $E_1$ is our desired solution profile provided that we can show that it satisfies $w \leq 0$ and $0 \leq v \leq 1$ and converges to $E_0$. It starts out heading southwest above the parabola $w' = 0$, then it meets the parabola and crosses, heading due west, at a point, then it heads northwest towards $E_0$. We will now show that it must converge to $E_0$ provided that (13) holds.

For $q > 0$, consider the right triangular region:

$$Q_q = \{(v, w) : w \leq 0, \ w + qv \geq 0, \ v \leq 1\}$$

which contains both equilibria. We show that if $q = c/2$, then $Q \equiv Q_{c/2}$ is positively invariant. No solution starting in it can leave it in positive time. The vector field along the $w = 0$ boundary of $Q$ points vertically down, into $Q$; along the boundary $v = 1$, the vector field points northwest, into $Q$. Note that the tangent line to the $w' = 0$ nullcline is $w = -\frac{r}{c}v$ and that the nullcline lies above its tangent line so, by (13), the portion of this nullcline with $w \leq 0$ lies in $Q$. Indeed, by (13), $r/c \leq c/4 < c/2$. 
Now we consider the portion of the boundary of \( Q \) where \( w + (c/2)v = 0 \). Compute:

\[
\frac{d}{dt}(w + lv) = w' + lv' \\
= (l-c)w - rv(1-v) \\
= l(c-l)v - rv(1-v), \text{ if } w + lv = 0, \\
> [l(c-l) - r]v, \ v > 0, \\
= [c^2/4 - r]v, \ l = c/2, \\
\geq 0, \ v > 0.
\]

The last inequality follows from (13). Therefore, on the line \( w = -(c/2)v \), the vector field points into \( Q \): if \( w(0) + (c/2)v(0) = 0 \), then \( w(t) + (c/2)v(t) > 0, t > 0 \). Therefore, \( Q \) is positively invariant.

This ensures that the left part of the unstable manifold of \( E_1 \) (green in figure) never leaves \( Q \) and therefore it must converge to \( E_0 \).

We have proved:

**Theorem 7.** For every \( c > 0 \) satisfying (13), there exists a strictly decreasing profile \( v \) satisfying (7) and (11), unique up to translation. For every \( c < 0 \) satisfying (13), there exists a strictly increasing profile \( w(s) = v(-s) \) satisfying (7), unique up to translation.

The second assertion is easily seen to follow from the first one.

We note that if \( v \) satisfies (7) and (11), then \( u(x,t) = v(a \cdot x - ct) \), where \( a, x \in \mathbb{R}^n \), is a plane wave solution of

\[
u_t = \Delta u + ru(1-u).
\]
8. **How big should a wild-life preserve be**

Consider a habitat $0 < x < L$ with lethal boundaries at both ends. Can a motile organism persist in such a habitat? Let $u$ represent the fractional density of the organism. Then $u = u(x,t)$ satisfies the equation

\[ u_t = du_{xx} + ru(1-u), \ x \in (0,L), \ t > 0. \]  

with "lethal boundary conditions":

\[ u(0,t) = u(L,t) = 0, \ t > 0. \]

The intrinsic growth rate is $r > 0$ and the motility is $d > 0$. To see why these boundary conditions represent "death at the boundary", keep in mind that we consider only non-negative solutions $u$ and hence for all $t > 0$ we must have $u_x(0,t) \geq 0$ and $u_x(L,t) \leq 0$. The total (fractional) density $U(t) = \int_0^L u(x,t)\,dx$ satisfies

\[ \frac{dU}{dt} = du_x(L,t) - du_x(0,t) + \int_0^L ru(1-u)\,dx. \]

So the first two terms represent loss at each boundary while the integral gives reproduction and death in the interior. If, instead of lethal boundary conditions, we had required that $u_x = 0$ at $x = 0, L$, then there would be no loss at the boundary. This would result if we place a fence around the wildlife refuge allowing no animals to leave. We are thinking of the wolves in Yellowstone National Park. If they stray outside park boundaries, chances are good that they are shot or in some way lost.

We ask if there can be a positive equilibrium solution. Then $u = u(x) > 0$ would satisfy

\[ u'' + mu(1-u) = 0, \ u(0) = u(L) \]

where $m = r/d$ and we write $u' = u_x$.

We may write (15) as a first order system

\[ \begin{align*}
    u' &= v \\
    v' &= -mu(1-u)
\end{align*} \]

It has equilibria $E_0 = (0,0)$ and $E_1 = (1,0)$. $E_0$ is a center with purely imaginary eigenvalues $\pm i\sqrt{m}$ and $E_1$ is a saddle point.

Equation (16) is a conservative system. Ignoring the boundary conditions, multiply (15) by $v = u'$ and integrate to find that

\[ v^2 + m(u^2 - \frac{2u^3}{3}) = \text{constant} \]
is constant with respect to $x$. Orbits are depicted below.

There is a homoclinic orbit surrounding $E_0$ with limit $E_1$ given by:

$$v^2 + mF(u) = mF(1) = m/3$$

where $F(u) \equiv u^2 - \frac{2u^3}{3}$. Observe that $F$ is strictly increasing on $(0, 1)$. Surrounding $E_0$, but inside the homoclinic orbit, are periodic (closed) orbits given by:

$$v^2 + mF(u) = mF(s)$$

for $0 < s < 1$. The portions of these orbits satisfying $u > 0$ are potential solutions of our boundary value problem (15). A solution $(u(x), v(x))$ starting at $(0, v_0)$ at $x = 0$, where $u = 0$, arrives at $(u_m, 0)$ at $x = L/2$, where $u$ is maximum, and continues, symmetrically with respect to the $u$-axis to point $(0, -v_0)$ at $x = L$, where $u = 0$ again. Here, $v_0^2 = mF(s)$ and $u_m = s$, for some $s \in (0, 1)$. Given $L > 0$, we must find such an $s$.

One sees that $u(x)$ is strictly increasing in $x$ for $0 < x < L/2$ with $u(0) = 0$ and $u(L/2) = s$ for, hopefully, some $s$, so

$$\frac{dx}{du} = \frac{1}{\frac{du}{dx}} = \frac{1}{v} = \frac{1}{\sqrt{m} \sqrt{F(s) - F(u)}}$$

![Figure 1. Orbits of (16) with $m = 3$.](image-url)
Integrating from $u = 0$ to $u = s$, we find that

$$L \sqrt{\frac{m}{2}} = \int_0^s \frac{du}{\sqrt{F(s) - F(u)}}$$

Since

$$F(s) - F(u) = s^2 - u^2 + (2/3)(u^3 - s^3) \leq (s^2 - u^2)$$

we have the inequality

$$L \sqrt{\frac{r}{d}} = L \sqrt{\frac{m}{2}} \geq \int_0^s \frac{du}{\sqrt{s^2 - u^2}} = \frac{\pi}{2}$$

or

$$L \geq \pi \sqrt{\frac{d}{r}}.$$  

This sets a lower limit on the size of a wild-life preserve having lethal boundary for an organism with motility $d$ and intrinsic growth rate $r$ to exist at equilibrium. Observe that highly motile species need larger refuges, as do slow growing ones.

Define $G(s) = \int_0^s \frac{du}{\sqrt{F(s) - F(u)}}$ for $0 < s < 1$. The change of variables $u = \eta s$ gives

$$G(s) = \int_0^1 \frac{s d\eta}{\sqrt{F(s) - F(s\eta)}} = \int_0^1 \frac{d\eta}{\sqrt{1 - \eta^2 + 2/3(\eta^3 - 1)s}}$$

One may check that the integrand is strictly increasing in $s$, implying that $G$ is strictly increasing in $s \in (0, 1)$. Moreover, the final expression is seen to imply that the limit

$$G(0+) = \int_0^1 \frac{d\eta}{\sqrt{1 - \eta^2}} = \frac{\pi}{2},$$

A bit of algebra establishes that for $s = 1$

$$G(1) = \int_0^1 \frac{3d\eta}{(1 - \eta)\sqrt{2\eta + 1}} \geq \frac{3}{\sqrt{3}} \int_0^1 \frac{d\eta}{(1 - \eta)} = +\infty$$

We conclude that the range of $G$ is the interval $(\pi/2, \infty)$.

In conclusion, for each value of $L \sqrt{m}/2 \in (\pi/2, \infty)$ there is a unique $s \in (0, 1)$ such that $G(s) = L \sqrt{m}/2$. This $s$ is precisely the maximum value of the population density $u$.

**Theorem 8.** If $L \leq \sqrt{d/r}$, there does not exist a positive equilibrium solution of (15); If $L > \sqrt{d/r}$, there is a unique positive equilibrium solution and it is symmetric about $x = L/2$ where it attains its maximum value, which is smaller than one. The maximum value increases with $L$ and limits at one as $L \to \infty$. 


REFERENCES