Math 300 Fall 2003 Final Exam December 12, 2003 Dr. Michelle Zandieh

Directions: 100 points total (+ 6 possible bonus pts.). The score will be scaled to 150 points. Please do not write on this sheet (except to write your name).

1. (15 pts.) Find the mistakes in the following proofs. For each mistake BRIEFLY explain why it is a mistake.

   a. (3 pts.) Find 1 mistake for full credit. Find a second mistake for up to 3 bonus points.

Statement: Let A and B be sets. If $A \cap B \subseteq A \cup B$, then $A \subseteq B$.

Proof: Let $x \in A \cap B$. Then $x \in A$ and $x \in B$ by definition of intersection. Since $x \in A$, by definition of union $x \in A \cup B$. Therefore, by definition of union, $x \in A$ or $x \in B$.

Case 1: $x \in A$. Since $A \subseteq B$, we know $x \in B$.

Case 2: $x \in B$

In either case, $x \in B$

Therefore, $A \subseteq B$ √

   b. (12 pts.) Find 4 mistakes for full credit. Find an additional mistake for up to 3 bonus points.

Statement: Let A, B, and C be sets. If $A \subseteq B$, then $C \backslash A \subseteq C \backslash B$

Proof: Suppose $C \backslash B$. Therefore $x \in C$ and $x \in \neg B$, by definition of set difference. Since $x \notin B$, and $A \subseteq B$, then $x \notin A$. So $x \in C$ and $x \in \neg A$. So by definition of set difference, $x \in C \backslash A$.

Therefore $C \backslash A \subseteq C \backslash B$ √

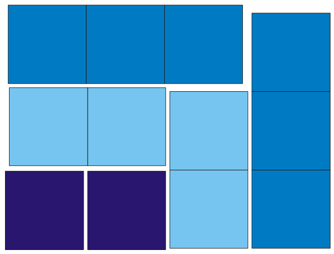
2. (22 pts.) Let f: $\boldsymbol{R} \rightarrow \boldsymbol{R}$ be defined by $f(x) = 3x - 4$.

   a. (4 pts.) What is the inverse, g, of f ?
   b. (6 pts.) Prove that g is an inverse of f.
   c. (6 pts.) Prove that g is 1-1.
   d. (6 pts.) Prove that g is onto.

3. (10 pts.) Find a function f: $\boldsymbol{N} \rightarrow \boldsymbol{Z}$ with the given properties.

   1. (5 pts.) 1-1 and onto. (Do NOT Prove f is 1-1 and onto)
   2. (5 pts.) 1-1 but not onto. (Do NOT Prove f is 1-1 but not onto)

4. (15 pts.) Define a ***k*-train** to be a row of k unit squares. The figure below shows that a 3×4 rectangle can be tiled using two 1-trains, two 2-trains, and two 3-trains:



Prove that for any $n\in\boldsymbol{N}$, an $n\times(n+1)$ rectangle can be tiled using exactly two of each train from size $k = 1$ to size $k = n$.

5. (18 pts.) Consider the following equivalence relation on $\boldsymbol{R}^2$:

$$(x, y) \equiv (u, v) \Leftrightarrow x - y^2 = u - v^2$$

a. (4 pts.) Prove that $\equiv$ is symmetric.

The relation $\equiv$ may be used to partition $\boldsymbol{R}^2$ into a family of sets, $\boldsymbol{F}$. When this is done, two points in $\boldsymbol{R}^2$ are in the same set $\boldsymbol{A} \in \boldsymbol{F}$ if and only if they are equivalent.

b. (4 pts.) Describe the sets in the partition algebraically.
c. (4 pts.) Describe the sets in the partition geometrically. (Include a picture.)
d. (6 pts.) Prove that $\bigcup_{A\in F} A = \boldsymbol{R}^2$.

6. (10 pts.) Let A and B be sets. Prove the following statement either by *contradiction* OR *contraposition*: If $A\cup B \subseteq B$ then $A \subseteq B$.

7. (10 pts.) For each $n\in\boldsymbol{N}$, let $B_n = \begin{cases} [-1-\frac{1}{n}, 1+\frac{1}{n}] \text{ if } n > 3 \\ \boldsymbol{N} \text{ if } n \le 3 \end{cases}$

a. (4 pts.) What is $\bigcap_{n=1}^{\infty} B_n$?

b. (6 pts.) What is $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} B_n$?