

1. Prove that the function $f : \mathfrak{R} \rightarrow \mathfrak{R}$ defined by $f(x) = x^3 - 1$ is one-to-one.

Pf: Let $x_1, x_2 \in \mathfrak{R}$ such that $f(x_1) = f(x_2)$. Then to prove that f is 1-1 I must show $x_1 = x_2$. Since the function is defined as $f(x) = x^3 - 1$ and $f(x_1) = f(x_2)$ then by transitive property $x_1^3 - 1 = x_2^3 - 1$. Since adding one to each side does not change their equality, $x_1^3 = x_2^3$. By taking the cube root of each side we get $\sqrt[3]{x_1^3} = \sqrt[3]{x_2^3} \Rightarrow x_1 = x_2$. Therefore f is one to one.

2. Suppose $f : A \rightarrow B$ is a function and $A_1, A_2 \subseteq A$. Prove the following, or disprove it by providing a counterexample. If the claim is not true, is at least one direction of subset inclusion true.

a) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$

Pf: This equality is false. One direction is true. (See proof below) A counterexample that works well is to let A be the set of integers, and B be the set of whole numbers then the function $f(x) = x^2$ is a function from $A \rightarrow B$. Let $A_1 = \{1, 2, 3, 4\}$ and $A_2 = \{-1, -2, 3, 4\}$, then $A_1 \cap A_2 = \{3, 4\}$ and $f(A_1 \cap A_2) = \{9, 16\}$. However, $f(A_1) = \{1, 4, 9, 16\}$ and $f(A_2) = \{1, 4, 9, 16\}$. So $f(A_1) \cap f(A_2) = \{1, 4, 9, 16\}$ which is not equal to $\{9, 16\}$ since $f(A_1 \cap A_2) \supseteq f(A_1) \cap f(A_2)$ is not true.

It is true though that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

Pf: Let $y \in f(A_1 \cap A_2)$ then since f is a function $\exists x \in A_1 \cap A_2$ such that $f(x) = y$. Then by definition of intersection $x \in A_1$ and $x \in A_2$. Thus $f(x) \in f(A_1)$ and $f(x) \in f(A_2)$. So by definition of intersection $f(x) \in f(A_1) \cap f(A_2)$. Since $f(x) = y$, then $y \in f(A_1) \cap f(A_2)$. So $y \in f(A_1 \cap A_2)$ implies $y \in f(A_1) \cap f(A_2)$. Thus $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ by definition of subset.

3. Suppose $f : A \rightarrow B$ is one-to-one. Show that $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

Pf: We show equality using subsets. The first direction was shown in 2a. So we need only to show that $f(A_1 \cap A_2) \supseteq f(A_1) \cap f(A_2)$. Let $y \in f(A_1) \cap f(A_2)$. Then $y \in f(A_1)$ and $y \in f(A_2)$ by definition of intersection. So $\exists x_1 \in A_1$ such that $f(x_1) = y$ and $\exists x_2 \in A_2$ such that $f(x_2) = y$. Since f is 1-1, then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. So $x \in A_1$ and $x \in A_2$ thus $x \in A_1 \cap A_2$ by definition of intersection. So $f(x) \in f(A_1 \cap A_2)$, but $f(x) = y$. Thus $y \in f(A_1 \cap A_2)$ and by the definition of subsets $f(A_1 \cap A_2) \supseteq f(A_1) \cap f(A_2)$.

4. Consider the function $f: \mathbf{R}$ to \mathbf{R} given by $f(x) = ax + b$.

a) Give conditions on a and b such that f is onto.

Ans: Since f maps \mathbf{R} to \mathbf{R} , then a and b must both be real numbers. Also in order for f to be onto $a \neq 0$. If $a = 0$ then $f = b$ which is a horizontal line, which is not onto.

b) Prove that f is onto if a and b satisfy these conditions.

Pf: Let y be a real number, then define $x = \frac{y-b}{a}$ so x is a real number since a, b are real and the reals are closed under division and subtraction. Then consider $f(x) = f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right) + b = y - b + b = y$. Thus for every real number y there is a real number x defined by $x = \frac{y-b}{a}$ such that $f(x) = y$. Therefore f is onto.

5. Define a function $f: \mathbf{N} \rightarrow \mathbf{Z}$ that is one-to-one and onto.

Answers may vary, here is an example.

$$\text{Let } f(x) = \begin{cases} \frac{-x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$$

a. Prove that f is a function.

Pf: To prove that f is a function I must show F1) and F2) hold.

F1) Let x be a natural number. Then either x is even or odd.

Case 1: If x is even then let $y = \frac{-x}{2}$. Since x is even then it is a multiple of 2 and thus $\frac{-x}{2}$ is an integer.

Case 2: If x is an odd number then let $y = \frac{x-1}{2}$. Since x is odd then

$x = 2k + 1$ for some natural number k . So $y = \frac{(2k+1)-1}{2} = \frac{2k}{2} = k$, which is a natural number and hence is an integer.

Therefore in either case for every given x there is an integer y such that $f(x) = y$.

F2) Let $f(x) = y_1$ and $f(x) = y_2$ then I must show that $y_1 = y_2$.

Again there are two cases, either x is even or x is odd.

Case 1: If x is even, then $y_1 = \frac{-x}{2}$ and $y_2 = \frac{-x}{2}$. Thus

$-2y_1 = x$ and $-2y_2 = x$. By the transitivity of equality $-2y_1 = -2y_2$. So $y_1 = y_2$.

Case 2: If x is odd the similarly $y_1 = \frac{x-1}{2}$ and $y_2 = \frac{x-1}{2}$. Thus

$2y_1 + 1 = x$ and $2y_2 + 1 = x$. By the transitivity of equality $2y_1 + 1 = 2y_2 + 1$.

So by rules of algebra $y_1 = y_2$.

Therefore since F1 and F2 hold f is a function.

b. Prove that it is one-to-one.

Pf: Suppose $x_1, x_2 \in \mathbf{N}$ such that $f(x_1) = f(x_2)$. Then $f(x_1)$ is either negative or non-negative.

Case 1: Suppose $f(x_1) = f(x_2)$ is non-negative. Then by the definition of the function $\frac{x_1-1}{2} = \frac{x_2-1}{2} \Rightarrow 2(x_1-1) = 2(x_2-1) \Rightarrow x_1 = x_2$ by rules of algebra.

Case 2: Suppose $f(x_1) = f(x_2)$ is negative. Then by the definition of the function $\frac{-x_1}{2} = \frac{-x_2}{2} \Rightarrow -2x_1 = -2x_2 \Rightarrow x_1 = x_2$ by rules of algebra.

In either case $x_1 = x_2$, therefore f is one-to-one.

c. Prove that it is onto.

Pf: Pick $y \in \mathbf{Z}$ then either y is positive, zero or negative.

Case 1: Suppose y is positive then let $x = 2y+1$, so x is odd and by the closure of addition and multiplication of the integers x is a positive integer, thus x is a natural number. Thus $f(x) = \frac{(2y+1)-1}{2} = \frac{2y}{2} = y$, so $x \in \mathbf{N}$ exists such that $f(x)=y$.

Case 2: Suppose y is zero then let $x=1$. Clearly 1 is a natural number. Since x is odd then $f(x) = \frac{(1)-1}{2} = 0$, so $x \in \mathbf{N}$ exists such that $f(x)=0$.

Case 3: Suppose y is negative then let $x = -2y$. Since y is negative, and the integers are closed under multiplication, then x is a positive integer, thus x is a natural number. Since x is a multiple of 2, x is even. Then $f(x) = \frac{-(-2y)}{2} = y$, so $x \in \mathbf{N}$ exists such that $f(x)=y$.

