MAT 300 Spring 06
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Homework #8
Solutions

1. Consider the family of sets consisting of all parabolas of the form $y = x^2 + a$ where $a$ is a real number.

a. Prove that this set of parabolas forms a partition of the plane, $\Re^2$.

Let $a \in \Re$ and let $A_a = \{(x, y) \in \Re^2 \mid y = x^2 + a\}$. Define the family $F = \{A_a\}_{a\in\Re}$. We wish to prove that $F$ is a partition of $\Re^2$.

(i) First, note that $(0, a) \in \Re^2$ since $0 \in \Re$ and $a \in \Re$. Also, $a = 0^2 + a$, so $(0, a) \in A_a$. Thus, $\forall A_a \in F$, $A_a \neq \phi$. So, $F$ satisfies the first property of a partition.

(ii) Next, suppose $A_a, A_b \in F$ and $A_a \cap A_b \neq \phi$. Then there exists $(x, y) \in A_a \cap A_b$ by the negation of the definition of the empty set. Thus, $(x, y) \in A_a$ and $(x, y) \in A_b$ by definition of intersection. By definition of $A_a$, $y = x^2 + a$, or $y - x^2 = a$. Similarly, $y = x^2 + b$ or $y - x^2 = b$ by definition of $A_b$. So, by the symmetric and transitive properties of equality, $a = y - x^2 = b$, so $a = b$. Thus, $A_a = A_b$, and $F$ satisfies the second property of a partition.

(iii) Finally, we wish to show that $\bigcup_{a\in\Re} A_a = \Re^2$. So, first let $(x, y) \in \bigcup_{a\in\Re} A_a$. Then $(x, y) \in A_a$ for some $a \in \Re$ by definition of the union of a family of sets. Thus, by definition of $A_a$, $(x, y) \in \Re^2$. Now, let $(x, y) \in \Re^2$. Consider the set $A_{y-x^2}$: note that $y = x^2 + (y - x^2)$, so $(x, y) \in A_{y-x^2}$. Since $\Re$ is closed under subtraction and multiplication, $y - x^2 \in \Re$, so $A_{y-x^2} \in F$. Thus, $(x, y) \in \bigcup_{a\in\Re} A_a$ by the definition of the union of a family of sets. So, $(x, y) \in \bigcup_{a\in\Re} A_a$ if and only if $(x, y) \in \Re^2$, and we have that $\bigcup_{a\in\Re} A_a = \Re^2$ by definition of set equality. Thus, $F$ satisfies the third property of a partition.

Since $F$ satisfies all three properties, $F$ is a partition of $\Re^2$.

b) Write an equivalence relation that will partition the plane into parabolas of the form $y = x^2 + a$ where $a$ is a real number.

We want two points $(x, y)$ and $(u, v)$ in $\Re^2$ to be equivalent if and only if they are on the same parabola, i.e. $y = x^2 + a$ and $v = u^2 + a$. In other words, we want $y - x^2 = a$ and

$v - u^2 = a$, or $y - x^2 = v - u^2$. So, define an equivalence relation $(x, y) \equiv (u, v)$ if and only if $y - x^2 = v - u^2$.

c) Prove that your answer in part b. is an equivalence relation.
We need to show that $\equiv$ is reflexive, symmetric, and transitive.

(i) Note first that, by the reflexive property of equality, $y - x^2 = y - x^2$ for any $(x, y) \in \Re^2$, so $(x, y) \equiv (x, y)$ and $\equiv$ is reflexive.

(ii) Next, let $(x, y), (u, v) \in \Re^2$, and suppose that $(x, y) \equiv (u, v)$. Then $y - x^2 = v - u^2$, and by the symmetric property of equality, $v - u^2 = y - x^2$. Thus, $(u, v) \equiv (x, y)$, and $\equiv$ is symmetric.

(iii) Finally, let $(x, y), (u, v)$, and $(r, s) \in \Re^2$, and suppose $(x, y) \equiv (u, v)$ and $(u, v) \equiv (r, s)$. Then $y - x^2 = v - u^2$ and $v - u^2 = s - r^2$ by definition of $\equiv$. So, by the transitive property of equality, $y - x^2 = s - r^2$, and thus $(x, y) \equiv (r, s)$ by definition of $\equiv$. So, $\equiv$ is transitive.

Since $\equiv$ is reflexive, symmetric, and transitive, it is an equivalence relation.