MAT 300 Spring 06
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Homework #7
Partial Solutions

Part I: Induction Proofs

a) Use the method of induction to prove $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$.

Pf: Let n = 0, then $\sum_{k=0}^{0} 2^k = 2^0 = 1$ and $2^{0+1} - 1 = 2 - 1 = 1$. Thus $\sum_{k=0}^{0} 2^k = 2^{0+1} - 1$. Now assume for n ≥ 1 that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$. Now $\sum_{k=0}^{n+1} 2^k = \sum_{k=0}^{n} 2^k + 2^{n+1} = (2^{n+1} - 1) + 2^{n+1}$ by our inductive hypothesis and $(2^{n+1} - 1) + 2^{n+1} = 2(2^{n+1}) - 1 = 2^{(n+1)+1} - 1$. Thus assuming the equality holds for n implies the inequality holds for n+1 and thus by the principle of mathematical induction for all n ≥ 1, $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$.

b) Note that if n$\ge$3, then $n^2 = n \cdot n \ge 3n = 2n + n > 2n + 1$.
Use this and induction to prove $2^n > n^2$ for all n ≥ 5.

Pf: Let n = 5, then 2^5 =32 and 5^2=25. Thus $2^5 > 5^2$.
Now assume for n ≥ 5 that $2^n > n^2$. Consider n+1. From the hint we know that $n^2 = n \cdot n \ge 3n = 2n + n > 2n + 1$, so $2^{n+1} = 2(2^n)$ which by the inductive hypothesis is greater than $2(n^2)$. Thus we have the following inequality holds:

$$2^{n+1} = 2(2^n) > 2(n^2) = n^2 + n^2 = n^2 + n \cdot n \ge n^2 + 3n = n^2 + 2n + n > n^2 + 2n + 1 = (n+1)^2$$

Thus $2^{n+1} > (n+1)^2$. by the inductive hypothesis Hence assuming the inequality holds for n implies the inequality holds for n+1 and thus by the principle of mathematical induction for all n ≥ 5, $2^n > n^2$.

Part III: Let $\equiv$ be the equivalence relation on **Z** given by: m$\equiv$n $\Leftrightarrow$ (m and n are the same distance from 8 on the number line).

a) Completely describe the partition of **Z** given by $\equiv$.

The partition can be described as the following family of sets

$$A_a = \{x \in Z : a = | x - 8 |\}$$

Notice there are other correct ways to describe this partition, which are equivalent to the family of sets above.

b) Prove that your answer to a) is a partition of **Z**.

To prove that the family of sets denoted by A_a forms a partition of **Z** we must show that each set in the partition is non-empty (i.e. $\forall a, A_a \neq \vartheta$), that if two sets in the partition overlap then they are equal (i.e. $A_a \cap A_b \neq \vartheta \Leftrightarrow a = b$), and that the union of the entire family of sets is equal to **Z** (i.e. $\cup A_a = Z$).

Pf: Let A_a, be any set in the family which describes the partition. Consider the element a +8. Since a and 8 are both integers and the set of integers is closed under addition then a+8 is an integer and |a+8-8| =a therefore by definition a+8 is an element of A_a $\forall a \in Z$. Therefore every set in the partition is non-empty.

Now assume that $A_a \cap A_b \neq \vartheta$ then there is an element x such that $x \in A_a$ and $x \in A_b$ So by the definition of the partition this means that |x-8|=a and |x-8|=b. Since = is transitive then a=b.

Finally, I must show $\cup A_a = Z$.

First show $\cup A_a \subset Z$. Let x be an element in the union of the family of A_a. By definition of the set $x \in Z$. So by definition of subset $\cup A_a \subset Z$. Now show $\cup A_a \supset Z$. Let $x \in Z$ then consider the value of |x-8|. Since x and 8 are both integers we know that |x-8| is also an integer and thus $x \in A_{|x-8|}$. Therefore by definition of union of a family of sets x is an elements of the union of A_a. Hence $\cup A_a \supset Z$. So by definition of equality $\cup A_a = Z$.

Thus the sets defined in part a) are a partition.