

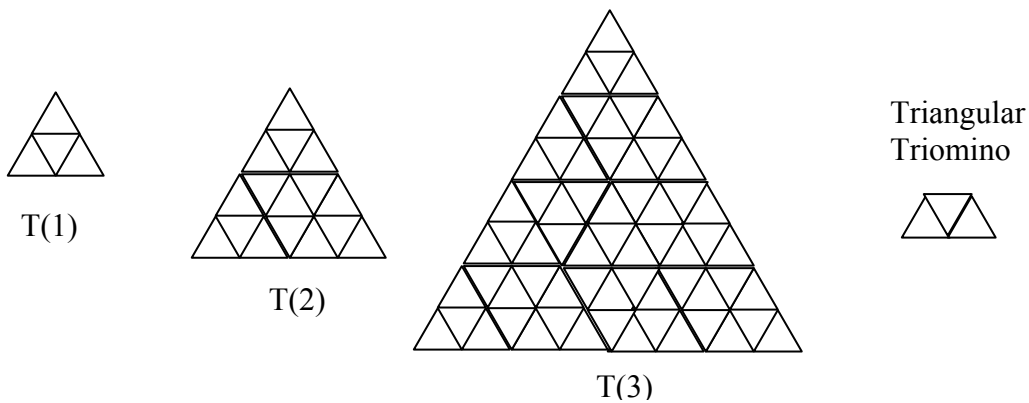
Part I: Recall the “L” shaped tiling of a checker board with a single square tile on one square from the board. Prove that if a 2^n by 2^n board has one square covered then the remaining board can be tiled with “L” shaped pieces. Remember that the “L” shaped pieces are made of 3 squares.

Pf: Base Step: Let $n = 1$. Consider the 2×2 board with one piece missing. Since there are three squares remaining, in each case the board can be tiled with a single L shaped piece.



Induction Step: Let $n \geq 1$. Assume that any 2^n by 2^n board with one square covered can be tiled with “L” shaped pieces. Then consider a 2^{n+1} by 2^{n+1} board with one square covered. Since $2(2^n) = 2^{n+1}$, the board can be divided into four $2^n \times 2^n$ boards. By dividing each side in half. The covered square is contained within one of these quadrants, thus by the inductive hypothesis this quadrant can be tiled with L-shaped pieces. Now place an L shaped piece in the center of the board such that it covers the corner square of each remaining quadrant. Now each of these has one square covered and by the inductive hypothesis can be tiled. So assuming that a 2^n by 2^n board with one square covered implies that a 2^{n+1} by 2^{n+1} board with one square covered can be tiled. Therefore by induction I have proved that any 2^n by 2^n board with one square covered can be tiled with “L” shaped pieces.

Part II: Consider a Triangular Board, $T(n)$, made up of equilateral triangles with side length 1 and 2^n triangles on each side. A triangular triomino is a piece made up of three triangles.

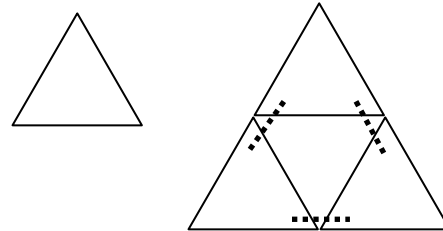


Prove: For any natural number $n \geq 1$, if any corner triangle is removed from $T(n)$ then the remaining board can be tiled with triangular triominos.

Pf: Let $n=1$ and consider $T(1)$ with one corner removed. Regardless of which corner is removed the remaining piece is a triomino, so it can be covered by a triomino.

Now let $n \geq 1$ and assume that $T(n)$ with one corner removed can be tiled with triominos. Consider $T(n+1)$ with a corner removed. Notice that $T(n+1)$ is made up of four $T(n)$ s, see pictures. Three of these $T(n)$ s have an exposed corner. So the missing corner can be any of the three.

By the induction hypothesis this triangle can be tiled with triominos. Now consider the other three $T(n)$ s. Regardless of which triangle is already tiled the remaining three have corners which touch opposite the corner which has been removed. See the picture.



Place a triomino such that it covers one triangle from each $T(n)$, then these now have one piece covered and by the induction hypothesis the remaining $T(n)$ can be tiled by triominos. Thus given that $T(n)$ can be tiled for any $n \geq 1$, then $T(n+1)$ can be tiled. Therefore by the principle of mathematical induction for any natural number $n \geq 1$, if any corner triangle is removed from $T(n)$ then the remaining board can be tiled with triangular triominos.

Part III: Consider checkerboards with dimensions $2 \times 2n$ and the checkerboard is made up of squares which are alternating colors, red and black. Prove that if one black and one red square are each covered by a single square tile on any $2 \times 2n$ board of this type, then it can be tiled by rectangular dominoes of size 2×1 .

Pf: Base Step: Let $n=1$ then the board is a 2×2 checkerboard with one black and one red square removed. Since the colors alternate then the remaining squares are adjacent and can therefore be covered by a domino.

Inductive Step: Let $n = k$ then assume a $2 \times 2k$ board with one red and one black square covered can be tiled with dominos. Now consider $n = k+1$. Then the checkerboard is 2 by $2(k+1) = 2k+2$. So it can be thought of as two adjacent boards of size 2 by $2k$ and 2×2 . There are three cases, either the covered squares are both in the 2 by $2k$ piece, or they are both in the 2×2 case or there is one in each.



Case 1: If both of the covered squares are in the 2 by $2k$ section of the board then by the inductive hypothesis this part can be tiled with dominos and the remaining 2×2 portion can have two dominos placed side by side. Therefore the $2 \times (2k+2)$ board can be tiled.

Case 2: If both of the covered squares are in the 2 by 2 section of the board then by the base step this part can be tiled with dominos and the remaining $2 \times 2k$ portion can have dominos placed side by side to tile the board. Therefore the $2 \times (2k+2)$ board can be tiled.

Case 3: Suppose one covered square is in each section of the board. Then place a domino adjacent to the covered square in the 2×2 piece so that it covers one square from each section of the board. (Notice the blue domino in the picture -- this would be the placement of the domino if either black square were covered.) Since the domino placed is adjacent to the covered square it covers a square of the alternate color. Therefore in the 2×2 section of the board one square of each color has been covered and by the base step the remaining squares of the 2×2 section can be tiled.



Notice the domino also covered a square of the opposite color in the $2 \times 2k$ section. Thus the $2 \times 2k$ section has two squares covered one of each color -- since the initial squares covered were different and the domino covers adjacent squares of which the colors alternate. Then by the inductive hypothesis the remaining section of the board can be tiled. Therefore the entire board has been tiled.

Thus in any case assuming that a $2 \times 2k$ checkerboard with one red and one black square covered can be tiled by dominos implies that a $2 \times 2(k+1)$ checkerboard with one red and one black square covered can be tiled by dominos. Therefore by the principle of mathematical induction if one black and one red square are each covered by a single square tile on any $2 \times 2n$ checker board, then it can be tiled by rectangular dominoes of size 2×1 .