MAT 300 Spring 06
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Homework #5 Solutions

Part 1: Let $\{A_n : n \in \mathbf{N}\}$ and be $\{B_n : n \in \mathbf{N}\}$ sequences of sets.

1. Prove:

   a. If $A_n \subseteq B_n\ \forall n \in \mathbf{N}$, then $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$.

   Proof: Let $x \in \bigcup_{n=1}^{\infty} A_n$. Then, by the definition of the union of a family of sets, there exists some $m \in \mathbf{N}$ such $x \in A_m$. Now, since $A_n \subseteq B_n\ \forall n \in \mathbf{N}$, we have that $A_m \subseteq B_m$. So, by the definition of subset, $x \in B_m$. Thus, by the definition of the union of a family of sets, $x \in \bigcup_{n=1}^{\infty} B_n$. So, $x \in \bigcup_{n=1}^{\infty} A_n$ implies that $x \in \bigcup_{n=1}^{\infty} B_n$, and by the definition of subset, $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$, as desired.

   b. If $A_n \subseteq B_n\ \forall n \in \mathbf{N}$, then $\bigcap_{n=1}^{\infty} A_n \subseteq \bigcap_{n=1}^{\infty} B_n$.

   Proof: Let $x \in \bigcap_{n=1}^{\infty} A_n$. Then, by the definition of the intersection of a family of sets, $x \in A_n$, $\forall n \in \mathbf{N}$. Now, since $A_n \subseteq B_n\ \forall n \in \mathbf{N}$, by the definition of subset, $x \in B_n$, $\forall n \in \mathbf{N}$. Thus, by the definition of the intersection of a family of sets, $x \in \bigcap_{n=1}^{\infty} B_n$. So, $x \in \bigcap_{n=1}^{\infty} A_n$ implies that $x \in \bigcap_{n=1}^{\infty} B_n$, and by the definition of subset, $\bigcap_{n=1}^{\infty} A_n \subseteq \bigcap_{n=1}^{\infty} B_n$, as desired.

2. Prove that $\left(\bigcap_{n=1}^{\infty} A_n\right)' \subseteq \bigcup_{n=1}^{\infty} A_n'$.

   Proof: Let $x \in \left(\bigcap_{n=1}^{\infty} A_n\right)'$. Then, by the definition of compliment, $x \notin \left(\bigcap_{n=1}^{\infty} A_n\right)$. So, by the (negation of the) definition of the intersection of a family of sets, there exists $m \in \mathbf{N}$ such $x \notin A_m$. But then, by the definition of compliment, $x \in (A_m)'$. So, by the definition of the union of a family of sets, x $\in \bigcup_{n=1}^{\infty} (A_n)'$. So, x $\in \left(\bigcap_{n=1}^{\infty} A_n \right)'$ implies x $\in \bigcup_{n=1}^{\infty} (A_n)'$, and by the definition of subset, $\left(\bigcap_{n=1}^{\infty} A_n \right)' \subseteq \bigcup_{n=1}^{\infty} (A_n)'$, as desired.

3. Suppose ***F*** = {A} is a family of sets, and suppose D is a set for which $D \subseteq A$ for every $A \in F$. Prove that $D \subseteq \cap_F A$.

   Proof: Let $d \in D$. Then since $D \subseteq A$ for every $A \in F$, $d \in A$ for every $A \in F$. Therefore by the definition of intersection $d \in \cap_F A$. Since $d \in D \to d \in \cap_F A$, then $D \subseteq \cap_F A$ by definition of subset.

Part 2: Consider the following statements:

a. $\forall\, n \in \boldsymbol{N}, \exists\, k \in \boldsymbol{R}$, such that $k>n$.
b. $\exists\, n \in \boldsymbol{N}$ such that $\forall\, k \in \boldsymbol{R}$, $k \le n$.

4. Explain what each of the statements (a. and b.) means. You must do more than just translate into words to get full credit.

   a. The statement in part a. says that, for all n in the natural numbers, there exists k in the real numbers such that k is greater than n. That is, the reals are not bounded by any natural number.

   b. The statement in part b. says that there exists an n in the natural numbers such that for all k in the real numbers, k is less than or equal to n. That is, there exists a natural number that is greater than all real numbers. (Notice, this is the negation of the statement in part a.)

5. Prove or disprove each statement (a. and b.).

   a. Let n$\in$**N** be an arbitrary natural number, and let k=n+1. Notice that k$\in$**N**, so clearly, k$\in$**R** and k>n. Thus, the statement is true for all n$\in$**N**.

   b. This statement is false. To disprove the statement, we need to show that no such n exists. That is, for all n$\in$**N**, there exists some k$\in$**R** such that k>n. But this is precisely the statement in part a., which we have already shown to be true (let k=n+1). So, we have shown the statement to be false.