MAT 300 Spring 06
Dr. Zandieh
Homework #4
Partial Solutions
Due March 9, 2006

2. (This is a problem that is SIMILAR to number 2)

Prove if $A \setminus B = A$, then $A \cap B = \phi$.

Proof: We will prove this statement by contradiction. Assume that $A \setminus B = A$ and $A \cap B \neq \phi$. By the negation of the definition of the empty set, $A \cap B \neq \phi$ means that there exists (at least one) $x \in A \cap B$. So, by the definition of intersection, $x \in A$ and $x \in B$. Since $x \in A$ and $A \setminus B = A$, we have that $x \in A \setminus B$ by the definition of set equality. Then $x \in A \cap B'$ by the definition of set difference. By the definition of intersection, we have $x \in A$ and $x \in B'$. Then $x \notin B$ by the definition of compliment. But we already showed that $x \in B$. **This is a contradiction** because we have that $x \notin B$ and $x \in B$, which cannot both be true. So, our assumption that $A \setminus B = A$ and $A \cap B \neq \phi$ was false. Thus, its negation (the original statement) must be true. In other words, we have shown that if $A \setminus B = A$, then $A \cap B = \phi$.

3a. (This is a problem SIMILAR to 3a)

$(A \cup B) \cap C = A \cup (B \cap C)$

This is a false statement. (Note that it is NOT the associative property!).
Counterexample: Let A = {1, 2, 3}, B = {1, 2}, and C = {1}.
Then $(A \cup B) \cap C$ = {1} but $A \cup (B \cap C)$ = {1, 2, 3}.√

3c. (This is a problem SIMILAR to 3c)

Prove $(A \cup B) \cup C = A \cup (B \cup C)$.

Proof: To show that $(A \cup B) \cup C = A \cup (B \cup C)$, we need to show two subset relationships: $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ and $(A \cup B) \cup C \supseteq A \cup (B \cup C)$.

We wish to show first that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$. Let $x \in (A \cup B) \cup C$. Then $x \in (A \cup B)$ or $x \in C$ by the definition of union. This brings about two cases:

Case 1: $x \in (A \cup B)$. Then, by the definition of union, $x \in A$ or $x \in B$. Again, this brings about two cases:

Case 1a: $x \in A$. Then it is true that $x \in A$ or $x \in (B \cup C)$, so $x \in A \cup (B \cup C)$ by the definition of union.

Case 1b: $x \in B$. Then $x \in (B \cup C)$ by the definition of union. So, we have that $x \in A$ or $x \in (B \cup C)$, and thus $x \in A \cup (B \cup C)$ by the definition of union.

Case 2: $x \in C$. Then by the definition of union, $x \in (B \cup C)$. So, we have that $x \in A$ or $x \in (B \cup C)$. So, $x \in A \cup (B \cup C)$ by the definition of union.

In each case, we see that $x \in A \cup (B \cup C)$. So, $x \in (A \cup B) \cup C$ implies $x \in A \cup (B \cup C)$, and thus $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ by the definition of subset.

Now we wish to show that $(A \cup B) \cup C \supseteq A \cup (B \cup C)$. Let $x \in A \cup (B \cup C)$. Then $x \in A$ or $x \in (B \cup C)$ by the definition of union. Thus, we have two cases:

Case 1: $x \in A$. Then $x \in (A \cup B)$ by the definition of union. So, $x \in (A \cup B)$ or $x \in C$ is true, and thus $x \in (A \cup B) \cup C$ by the definition of union.

Case 2: $x \in (B \cup C)$. Then, by the definition of union, $x \in B$ or $x \in C$, so we have two cases:

Case 2a: $x \in B$. Then $x \in (A \cup B)$ by the definition of union. So, $x \in (A \cup B)$ or $x \in C$ is true, and thus $x \in (A \cup B) \cup C$ by the definition of union.

Case 2b: $x \in C$. Then it is true that $x \in (A \cup B)$ or $x \in C$, so $x \in (A \cup B) \cup C$ by the definition of union.

In each case, we see that $x \in (A \cup B) \cup C$. So, $x \in A \cup (B \cup C)$ implies $x \in (A \cup B) \cup C$, and thus $(A \cup B) \cup C \supseteq A \cup (B \cup C)$ by the definition of subset.

Since $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ and $(A \cup B) \cup C \supseteq A \cup (B \cup C)$, the definition of set equality gives us that $(A \cup B) \cup C = A \cup (B \cup C)$.

Alternate proof: We want to show that $(A \cup B) \cup C = A \cup (B \cup C)$. First, note that $x \in (A \cup B) \cup C$ is the same as $x \in (A \cup B)$ or $x \in C$ by the definition of union. Then we have $(x \in A$ or $x \in B)$ or $x \in C$, again by the definition of union. Now construct a truth table to determine the associative property of "or" which would mean the statements $(x \in A$ or $x \in B)$ or $x \in C$ and $x \in A$ or $(x \in B$ or $x \in C)$ are logically equivalent.

$x \in A$	$x \in B$	$x \in C$	$x \in A \vee x \in B$	$(x \in A \vee x \in B) \vee x \in C$	$x \in B \vee x \in C$	$x \in A \vee (x \in B \vee x \in C)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Note that the definition of union gives us that $x \in A$ or $(x \in B$ or $x \in C)$ is the same as $x \in A$ or $x \in (B \cup C)$, which in turn is equivalent to $x \in A \cup (B \cup C)$ again by the definition of union. The definition of set equality tells us that if $x \in (A \cup B) \cup C$ is equivalent to $x \in A \cup (B \cup C)$, then $(A \cup B) \cup C = A \cup (B \cup C)$. Since the columns corresponding to $(x \in A$ or $x \in B)$ or $x \in C$ and $x \in A$ or $(x \in B$ or $x \in C)$ are the same, the statements are logically equivalent. So, we have that $x \in (A \cup B) \cup C$ if and only if $x \in A \cup (B \cup C)$. So, by the definition of set equality, $(A \cup B) \cup C = A \cup (B \cup C)$.

5. Prove: A $\subseteq$ B if and only if A\B = $\varnothing$

Proof: "A $\subseteq$ B $\Rightarrow$ A\B = $\varnothing$" (Will do by contrapositive.)

Suppose that A\B $\neq \varnothing$. Then $\exists$ x such that x$\in$ A\B. Then by definition of set difference, there exists and x such that x$\in$A but x$\notin$B. But this means that A $\not\subset$ B.

So we have shown that A\B $\neq \varnothing \Rightarrow$ A $\not\subset$ B. Then by contraposition A $\subseteq$ B $\Rightarrow$ A\B = $\varnothing$.

"A\B = $\varnothing\Rightarrow$ A $\subseteq$ B" (Will do by contradiction.)

Assume that $A \setminus B = \phi$ and $A \not\subset B$. By the negation of the definition of subset, $A \not\subset B$ means that there exists (at least one) $x \in A$ such that $x \notin B$. Then $x \in B'$ by the definition of compliment. Since $x \in A$ and $x \in B'$, we have that $x \in A \cap B'$ by the definition of intersection. Thus $x \in A \setminus B$ by the definition of set difference. **This is a contradiction** because we assumed that $A \setminus B = \phi$, which means that $A \setminus B$ contains no elements, by the definition of the empty set. So, our assumption that $A \setminus B = \phi$ and

$A \not\subset B$ was false. Thus, its negation (the original statement) must be true. In other words, we have shown that if $A \setminus B = \phi$, then $A \subseteq B$.