Practice Test#2
MAT 271
Dr. Taylor's Class
Fall'08

Instructions: Give yourself one hour to work this practice exam. We will discuss in class on Monday

*Disclaimer: This practice test is intended only to help you assess your readiness for the midterm exam and should not be construed to imply that these or nearly identical problems will be on the actual midterm. The actual exam problems may resemble these not at all in terms of specific details. You need to study *all* the material we have covered.*

1. Compute the Fourier series of the function $f(x) = x^2$ on the interval $[-1, 1]$. (Hint: $\int x^2 \cos(nx)dx = \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}$) *Also it happens that* $\int x^2 \cos(n\pi x)dx = \frac{2x\cos(n\pi x)}{n^2\pi^2} + \frac{2\sin(n\pi x)}{n^3\pi^3}$*, so* $a_n = \int_{-1}^{1} x^2 \cos(n\pi x)dx = \frac{4(-1)^n}{n^2\pi^2}$*, and* $a_0 = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{3}$*. On the other hand,* x^2 *is an even function,* $\sin(n\pi x)$ *is an odd function, so* $b_n = 0$ *for all* $n > 0$*. From this it follows that the Fourier series is* $\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2}\cos(n\pi x)$.

2. Starting with your answer to the last question, use the fact that the $\frac{d}{dx}x^2 = 2x$ to compute the Fourier series of the function $g(x) = x$ on the interval $[-1, 1]$. *Differentiating the previous Fourier series term by term we arrive at* $\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi}\sin(n\pi x)$

3. Apply your knowledge of the geometric series to compute the Taylor series of the function $f(x) = \frac{1}{1+x^2}$ at $c = 0$, as well as its interval of convergence.*Since* $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$*, we have* $\frac{1}{1+y} = \sum_{n=0}^{\infty}(-1)^n y^n$*. The radius of convergence of these series is* $|y| < 1$*. From this it follows that* $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n(x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ *when* $x^2 < 1 \Leftrightarrow |x| < 1$.

4. On the basis of the last question, compute the Taylor series of the function $g(x) = \tan^{-1}(x)$. What is its radius of convergence? *Note that that* $\int_0^x \frac{1}{1+y^2}dy = \tan^{-1}(x)$*. We may integrate the series of the last question term by term to obtain* $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$*; the radius of convergence was* $|x| < 1$ *in the last question, and stays the same here.*

5. Determine the radius and interval of convergence:

$$\sum_{n=1}^{\infty} \frac{\tan^{-1}(k)}{k!}(x-2)^k$$

(hint: the $\tan^{-1}(k)$ has very little to do with it!) *First of all, for k sufficiently large* $\left|\frac{1}{k!}(x-2)^k\right| < \left|\frac{\tan^{-1}(k)}{k!}(x-2)^k\right| < \left|\frac{\pi}{2k!}(x-2)^k\right|$, *so that this series converges absolutely when* $\sum_{n=1}^{\infty} \frac{1}{k!}(x-2)^k$, *which converges to* $e^{x-2}-1$ *for all x; hence the radius of convergence is ∞ and the interval is* $(-\infty, \infty)$.

6. Determine whether the series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{k!}$$

Converges absolutely

7. Determine the convergence of the series

$$\sum_{k=1}^{\infty} \frac{k+1}{k^4+1}$$

We can use the limit comparison test on this series. Note that $\lim_{k\to\infty} \frac{\frac{k+1}{k^4+1}}{\frac{1}{k^3}} = 1$; *by the p-test with $p = 3$, this series converges.*