

Stochastic semigroups: their construction by perturbation and approximation

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Abstract. The main object of the paper is to present a criterion for the minimal semigroup associated with the Kolmogorov differential equations to be stochastic in ℓ^1 . Our criterion uses a weighted ℓ^1 -space. As an abstract preparation we present a perturbation theorem for substochastic semigroups which generalizes known results to the case of ordered Banach spaces which need not be AL -spaces. We also consider extensions of Kolmogorov's equations to spaces of measures. In an appendix we present a version of the Miyadera perturbation theorem for positive semigroups on ordered Banach spaces.

Key words: semigroup, substochastic, stochastic, perturbation, approximation, Kolmogorov differential equations

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1 Introduction

Let X be an ordered real Banach space with a generating (reproducing) cone X_+ , $X = X_+ - X_+$, and a norm which is additive on X_+ , $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in X_+$.

A C_0 -semigroup $(S(t)); t \geq 0$ (of bounded linear operators) on X is called *substochastic* (*stochastic*) if $S(t)$ is positive and $\|S(t)x\| \leq \|x\|$ ($\|S(t)x\| = \|x\|$) for all $x \in X_+$, $t \geq 0$.

The norm on X_+ can be uniquely extended to a positive bounded linear functional $\varphi \in X^*$ satisfying $\varphi x = \|x\|$ for all $x \in X_+$. The generator A of a stochastic semigroup necessarily satisfies $\varphi Ax = 0$ for all $x \in D(A) \cap X_+$.

It is the aim of this note to indicate conditions for a C_0 -semigroup defined by a limiting procedure to be stochastic. Differently from [17], [24; Sec. 2], [3] we work with an auxiliary Banach space X_1 that is continuously and densely embedded into X . This space may be of its own interest: in case of the Kolmogorov differential equations (Section 4), it can be chosen as the first moment space $\ell^{11} = \{x = (x_j)_{j=0}^\infty; \|x\|_1 := \sum_{j=0}^\infty (1+j)|x_j| < \infty\}$. By perturbation (Section 2) we will construct substochastic C_0 -semigroups on X which leave the auxiliary Banach space invariant and induce C_0 -semigroups thereon. These semigroups can also be obtained by an approximation procedure (Section 3). As a second application we present a measure-valued generalization of Kolmogorov's differential equations (Section 5) whose solutions are associated with Markov transition functions. In an appendix we present a version of the Miyadera perturbation theorem for positive semigroups on ordered Banach spaces. This result is needed in Section 2.

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2 Perturbation

A generating cone X_+ is always non-flat; cf. Remark A.1(a). The additivity of the norm on X_+ implies that the norm is monotone, and hence X_+ is normal; cf. [10; Proposition A.2.2]. The additivity of the norm on X_+ also implies that any bounded monotone sequence is convergent.

2.1 REMARKS. (a) Let S be a positive C_0 -semigroup on X , with generator A . We recall that then $D(A)_+ (= D(A) \cap X_+)$ is dense in X_+ . Moreover, S is substochastic (stochastic) if and only if $\varphi Ax \leq 0$ ($\varphi Ax = 0$) for all $x \in D(A)_+$. This follows immediately from the fact that, for $x \in D(A)_+$, one has

$$\frac{d}{dt} \|S(t)x\| = \frac{d}{dt} \varphi(S(t)x) = \varphi(AS(t)x).$$

Further, if D is a core for A , and $\varphi Ax = 0$ for all $x \in D$, then it follows that $\varphi Ax = 0$ for all $x \in D(A)$, and that therefore S is stochastic.

(b) Note that, although the norm of X is supposed to be additive on X_+ , we do not suppose X to be an AL -space. In fact, the order need not be associated with a Banach lattice structure. We want to present the abstract part of the paper in this general setting, although in the applications presented in Sections 4 and 5 the space X will be an AL -space. Nevertheless, we think that it is interesting to know that certain properties are valid in the general setting we present here.

2.2 Theorem. *Let S_0 be a positive C_0 -semigroup on X , with generator A_0 . Let $B: D(A_0) \rightarrow X$ be a positive linear operator satisfying*

$$\varphi(A_0 + B)x \leq 0 \quad (x \in D(A_0)_+).$$

(a) *Then, for all $x \in D(A_0)_+$, one has*

$$\int_0^\infty \|BS_0(s)x\| ds \leq \|x\|.$$

For all $0 \leq r < 1$, the operator $A_0 + rB$ is the generator of a substochastic C_0 -semigroup S_r . For $0 \leq r \leq r' < 1$ one has $S_r(t) \leq S_{r'}(t)$ ($t \geq 0$).

(b) *$S(t) := \text{s-lim}_{r \rightarrow 1} S_r(t)$ exists, uniformly for t in bounded subsets of $[0, \infty)$. S thus defined is a substochastic C_0 -semigroup on X . The generator of S is an extension of $A_0 + B$.*

(c) *S is the smallest positive semigroup (also called the minimal semigroup) whose generator is an extension of $A_0 + B$.*

Proof. (a) Since $B: D(A_0) \rightarrow X$ is positive we obtain that $B(\lambda - A_0)^{-1}$ is positive, hence bounded (cf. [10; Proposition A.2.11]), i.e., B is A_0 -bounded. Moreover, for $0 \leq t < \infty$, we obtain

$$\begin{aligned} \int_0^t \|BS_0(s)x\| ds &= \int_0^t \varphi(BS_0(s)x) ds \\ &\leq - \int_0^t \varphi(A_0 S_0(s)x) ds = -\varphi \int_0^t A_0 S_0(s)x ds \\ &= \varphi(x - S_0(t)x) = \|x\| - \|S_0(t)x\| \leq \|x\| \end{aligned}$$

and hence

$$\int_0^\infty \|BS_0(s)x\| ds \leq \|x\|,$$

for all $x \in D(A_0)_+$ (cf. [24; Lemma 1.2(a)]).

Using Theorem A.2, one proves the remaining assertions in the same way as in [24; Lemma 1.2 and Proposition 1.4]. \square

2.3 REMARK. The first version of a result as in Theorem 2.2 is due to Kato [17]. The application of the Miyadera perturbation theorem in this context goes back to [24]. Further developments can be found in [3], [15], [7], [6].

Next, we show a result on invariant subspaces of the dual semigroup. This result will be used in Section 5.

2.4 Proposition. *Let the assumptions of Theorem 2.3 be satisfied, and let Y be a weakly* sequentially closed subspace of X^* which is invariant under $((\lambda - A_0)^{-1})^*$ and $(B(\lambda - A_0)^{-1})^*$. Then Y is invariant under the semigroups S_r^* and S^* , with S_r and S from Theorem 2.3.*

Proof. It follows from the proof of Theorem A.2 that the spectral radius of $F(\lambda) = B(\lambda - A_0)^{-1}$ does not exceed one. Set $A_r = A_0 + rB$. Then

$$(\lambda - A_r)^{-1} = (\lambda - A_0)^{-1} \sum_{j=0}^{\infty} (rF(\lambda))^j.$$

Since Y is weakly* sequentially closed and invariant under $((\lambda - A_0)^{-1})^*$ and $F(\lambda)^*$, it is invariant under $((\lambda - A_r)^{-1})^*$. Since $S_r(t) = \text{s-lim}_{n \rightarrow \infty} (I - (t/n)A_r)^{-n}$, Y is invariant under $S_r^*(t)$, and also under $S^*(t)$ because $S(t)$ is the strong limit of $S_r(t)$. \square

If, in Theorem 2.2, one assumes $\varphi(A_0 + B)x = 0$ for all $x \in D(A_0) \cap X_+$, one cannot conclude that S is stochastic, in general; cf. [17; §4, Example 3]. Discussing conditions for this to hold has been a major objective in most of the pertinent work; cf. [17], [20], [24], [3], [8], [5], [4], [2], [6], [15]. Besides the development of the theory in the more general context, it was the main motivation of the present paper to indicate a new type of sufficient condition; cf. Theorem 2.7.

2.5 ASSUMPTION. Let X_1 be a subspace of X such that the following hold:

- There exists a norm $\|\cdot\|_1$ on X_1 which makes it a Banach space.
- $(X_1, \|\cdot\|_1)$ is continuously embedded into $(X, \|\cdot\|)$ and $X_1 \cap X_+$ is dense in X_+ .
- $X_{1,+} := X_1 \cap X_+$ is a generating cone for X_1 and $\|\cdot\|_1$ is additive on $X_{1,+}$.

It follows from these assumptions that there exists a unique positive functional $\varphi_1 \in X_1^*$ which coincides with $\|\cdot\|_1$ on $X_{1,+}$.

2.6 Proposition. *Let the assumptions of Theorem 2.2 be satisfied. Assume that X_1 is a subspace X which satisfies Assumption 2.5.*

Assume that S_0 induces a (necessarily positive) C_0 -semigroup \check{S}_0 on $(X_1, \|\cdot\|_1)$. Let \check{A}_0 denote its generator. (Note that \check{A}_0 is the restriction of A_0 to $D(\check{A}_0) = \{x \in D(A_0) \cap X_1; A_0x \in X_1\}$.) Assume that $B(D(\check{A}_0)) \subseteq X_1$, and assume that there is a constant $c > 0$ such that

$$\varphi_1(A_0 + B)x \leq c\|x\|_1 \quad (x \in D(\check{A}_0)_+).$$

Then the semigroup S from Theorem 2.2 leaves X_1 invariant and induces a positive C_0 -semigroup \check{S} on X_1 . \check{S} is the smallest positive C_0 -semigroup on X_1 whose generator is an extension of $\check{A}_0 + \check{B}$. For $x \in X_{1,+}$, $t \geq 0$, we have $\|S(t)x\| \leq e^{ct}\|x\|$, i.e., the rescaled semigroup $(e^{-ct}\check{S}(t))_{t \geq 0}$ is substochastic on X_1 .

Proof. Note first that the hypothesis can be reformulated as

$$\varphi_1((\check{A}_0 - c) + \check{B})x \leq 0, \quad (x \in D(\check{A}_0)_+),$$

where \check{B} denotes the restriction of B to $D(\check{A}_0)$. Therefore Theorem 2.2 can be applied to X_1 and the restricted operators.

Let $0 \leq r < 1$. Then Theorem 2.2(b) implies that $(\check{A}_0 - c) + r\check{B}$ is the generator of a substochastic semigroup on X_1 , or equivalently, that $\check{A}_0 + t\check{B}$ is the generator of a positive C_0 -semigroup \check{S}_r on X_1 satisfying $\|\check{S}_r(t)x\| \leq e^{ct}\|x\|$ for all $x \in X_{1,+}$, $t \geq 0$. It is easy to see that \check{S}_r and S_r (from Theorem 2.2) coincide on X_1 . Taking $r \rightarrow 1$ one obtains the assertion. \square

2.7 Theorem. *Let the assumptions of Proposition 2.6 be satisfied. Assume additionally that $-A_0$ is positive and that there exists $\varepsilon > 0$ such that (with $c > 0$ from Proposition 2.6)*

$$\varphi_1(A_0 + B)x \leq c\|x\|_1 - \varepsilon\|A_0x\| \quad (x \in D(\check{A}_0)_+),$$

i.e., for all $x \in D(A_0)_+ \cap X_1$ with $A_0x \in X_1$.

Then the generator \check{A} of \check{S} is a restriction of $A_0 + B$, and the generator A of S is the closure of \check{A} and of $A_0 + B$ in X .

If $\varphi(A_0 + B)x = 0$ for all $x \in D(A_0)_+$ then S is stochastic.

2.8 REMARK. The assumption that $-A_0$ is a positive operator can be replaced by the following:

If (x_n) is sequence in $D(\check{A}_0)_+$, $x_n \uparrow x \in X_1$ and $\sup_{n \in \mathbb{N}} \|A_0x_n\| < \infty$, then $x \in D(A_0)$.

Proof of Theorem 2.7. Let (r_n) be a sequence in $[0, 1)$, $r_n \uparrow 1$. We shall use the notation $A_n := A_0 + r_n B$ ($n \in \mathbb{N}$).

We know that $A_0 + B \subseteq A$ and $\check{A} \subseteq A$. In order to show the first assertion we thus have to show $D(\check{A}) \subseteq D(A_0)$.

Let $x \in X_{1,+}$, $\lambda > \max\{0, c\}$. Then $x_n := (\lambda - A_n)^{-1}x \in D(\check{A}_0)_+$ ($n \in \mathbb{N}$). By hypothesis,

$$\begin{aligned} \varphi_1(-x + \lambda x_n) &= \varphi_1 A_n x_n \leq \varphi_1(A_0 + B)x_n \leq c\|x_n\|_1 - \varepsilon\|A_0x_n\|, \\ \varepsilon\|A_0x_n\| &\leq (\lambda - c)\|x_n\|_1 + \varepsilon\|A_0x_n\| \leq \varphi_1 x = \|x\|_1. \end{aligned}$$

The sequence (x_n) is increasing and converges in X_1 to $(\lambda - A)^{-1}x$. Since A_0 preserves monotone sequences, (A_0x_n) is a bounded monotone sequence in X and thus has a limit in X . Since A_0 is a closed operator, $(\lambda - A)^{-1}x \in D(A_0)$. This implies $D(\check{A}) = (\lambda - A)^{-1}X_1 \subseteq D(A_0)$ since $X_{1,+}$ is a generating cone for X_1 .

The domain $D(\check{A})$ of \check{A} is a subset of $D(A)$, invariant under S ($= \check{S}$ on X_1) and dense in X , and therefore a core for A (cf. [19; Theorem X.49]). From $\check{A} \subseteq A_0 + B$ ($\subseteq A$) we also obtain $A = \overline{A_0 + B}$.

Now the last statement is a consequence of Remark 2.1(a). \square

3 Approximation

3.1 Proposition. *Let S_0, A_0, B, S be as in Theorem 2.2, in particular recall*

$$\varphi(A_0 + B)x \leq 0 \quad (x \in D(A_0)_+).$$

For $n \in \mathbb{N}$ let $B_n: D(A_0) \rightarrow X$ be a positive linear operator, $B_n x \leq Bx$ ($x \in X_+, n \in \mathbb{N}$), $B_n x \rightarrow Bx$ ($n \rightarrow \infty$) for all $x \in D(A_0)$. For $n \in \mathbb{N}$ let S_n be the smallest positive (substochastic) C_0 -semigroup whose generator is an extension of $A_0 + B_n$ (see Theorem 2.2(a)).

(a) *Then $S_n(t) \leq S(t)$ ($n \in \mathbb{N}_0$) and $S(t) = s\text{-}\lim_{n \rightarrow \infty} S_n(t)$ ($t \geq 0$).*

(b) *Assume additionally that X_1 is a subspace of X satisfying Assumption 2.5. As in Proposition 2.6, assume that S_0 induces a C_0 -semigroup \check{S}_0 on X_1 , and let \check{A}_0 denote its generator. Assume that $B_n(D(\check{A}_0)) \subseteq X_1$ ($n \in \mathbb{N}$), $B(D(\check{A}_0)) \subseteq X_1$, that $B_n x \rightarrow Bx$ in X_1 ($n \rightarrow \infty$) for all $x \in D(\check{A}_0)$, and that there is a constant $c > 0$ such that*

$$\varphi_1(A_0 + B)x \leq c\|x\|_1 \quad (x \in D(\check{A}_0)_+).$$

Then S_n ($n \in \mathbb{N}$) and S induce positive C_0 -semigroups \check{S}_n ($n \in \mathbb{N}$) and \check{S} on X_1 , respectively, and $\check{S}(t) = s\text{-}\lim_{n \rightarrow \infty} \check{S}_n(t)$ in X_1 ($t \geq 0$).

Proof. (a) (cf. [24; proof of Proposition 1.6]) For $0 \leq r < 1$ we obtain from Theorem 2.2(a) that the operators $A_0 + rB_n$ ($n \in \mathbb{N}$) and $A_0 + rB$ are generators of substochastic C_0 -semigroups $S_{n,r}$ ($n \in \mathbb{N}$) and S_r , respectively. The inequalities required in the hypothesis imply $S_{n,r}(t) \leq S_r(t)$ ($t \geq 0, n \in \mathbb{N}$). Taking $r \rightarrow 1$ we obtain $S_n(t) \leq S(t)$ ($t \geq 0, n \in \mathbb{N}$). A minor adaptation of [23; Theorem 1.4] to our context shows that $S_{n,r}(t) \rightarrow S_r(t)$ strongly ($n \rightarrow \infty$).

Let $x \in X_+$. The inequalities

$$\begin{aligned} 0 &\leq S(t)x - S_n(t)x = (S(t)x - S_r(t)x) + (S_r(t)x - S_{n,r}(t)x) + (S_{n,r}(t)x - S_n(t)x) \\ &\leq (S(t)x - S_r(t)x) + (S_r(t)x - S_{n,r}(t)x) \end{aligned}$$

imply $\|S(t)x - S_n(t)x\| \leq \|S(t)x - S_r(t)x\| + \|S_r(t)x - S_{n,r}(t)x\|$. Choosing first r close enough to 1 and then n large enough we can make the right hand side as small as we want. This implies that $S_n(t) \rightarrow S(t)$ strongly.

(b) Proposition 2.6 implies that the semigroups S_n ($n \in \mathbb{N}$) and S induce C_0 -semigroups \check{S}_n ($n \in \mathbb{N}$) and \check{S} on X_1 , and that these semigroups are the smallest positive semigroups on X_1 whose generators are extensions of $\check{A}_0 + \check{B}_n$ ($n \in \mathbb{N}$) and $\check{A}_0 + \check{B}$, respectively. Part (a), applied to these semigroups, yields $\check{S}(t) = s\text{-}\lim_{n \rightarrow \infty} \check{S}_n(t)$ ($t \geq 0$). \square

3.2 REMARK. Assuming monotonicity, i.e., $B_n x \leq B_{n+1} x$ ($x \in D(A)_+, n \in \mathbb{N}$), in Proposition 3.1, one could simplify the hypothesis in part (b). In this case the convergence $B_n x \rightarrow Bx$ in X_1 would follow from the remaining hypotheses.

4 Example: Kolmogorov's differential equations

The infinite system of differential equations

$$x'_j = \sum_{k=0}^{\infty} \alpha_{jk} x_k \quad (j = 0, 1, 2, \dots) \quad (4.1)$$

is known as Kolmogorov's differential equations provided the coefficients α_{jk} form a Kolmogorov matrix [16; Sec. 23.12], i.e.,

- for all $j, k \in \mathbb{N}_0$, $\alpha_{jk} \geq 0$ if $j \neq k$, $\alpha_{jj} \leq 0$,
- $\sum_{j=0}^{\infty} \alpha_{jk} = 0$ for all $k \in \mathbb{N}_0$.

x_j can be interpreted as the probability that the number of individuals in a population is j . As we allow the population to go extinct and be possibly resurrected by immigration, we choose the non-negative integers \mathbb{N}_0 as state space.

For $j \neq k$, α_{jk} is the rate at which the population size changes from k to j , while $-\alpha_{jj}$ is the rate at which a population of size j changes to a size different from j .

Typically Kolmogorov's differential equations are considered on the standard sequence space $\ell^1 = \{x = (x_j)_{j=0}^{\infty}; \|x\| < \infty\}$ with $\|x\| = \sum_{j=0}^{\infty} |x_j|$. See [17], [13; XVII.9], [14; XIV.7], [16; Chap. 23], [24], and the literature cited there. As $\sum_{j=1}^{\infty} jx_j$ is the expected population size, the first moment space $\ell^{11} = \{x = (x_j)_{j=0}^{\infty}; \|x\|_1 < \infty\}$ with $\|x\|_1 = \sum_{j=0}^{\infty} (1+j)|x_j| < \infty$ is also a meaningful state space. Interestingly enough, this will help us find a condition for the semigroup associated with (4.1) to be stochastic on ℓ^1 .

4.1 ASSUMPTION. There exist constants $c, \varepsilon > 0$ such that

$$\sum_{j=0}^{\infty} j\alpha_{jk} \leq c(1+k) - \varepsilon|\alpha_{kk}| \quad \text{for all } k \in \mathbb{N}_0.$$

Notice that $\sum_{j=0}^{\infty} j\alpha_{jk}$ can be interpreted as expected population growth rate at population size k . Let

$$D_0 = \{x = (x_j) \in \ell^1; \sum_{j=0}^{\infty} |\alpha_{jj}||x_j| < \infty\}.$$

4.2 Theorem. *Let (α_{jk}) be a Kolmogorov matrix which satisfies Assumption 4.1. Then the closure of the operator $A_{\infty}: D_0 \rightarrow \ell^1$,*

$$(A_{\infty}x)_j = \sum_{k=0}^{\infty} \alpha_{jk}x_k \quad (x = (x_k) \in D_0),$$

is the generator of a stochastic semigroup S on ℓ^1 .

The semigroup S leaves ℓ^{11} invariant and induces a strongly continuous semigroup \check{S} on ℓ^{11} . The generator of \check{S} is the restriction of A_{∞} to

$$D(\check{A}) = \{x \in \ell^{11} \cap D_0; A_{\infty}x \in \ell^{11}\}.$$

Moreover $\|S(t)x\|_1 \leq e^{ct}\|x\|_1$ for all $x \in \ell^{11}$, $t \geq 0$.

This theorem is a consequence of Theorem 2.2, Proposition 2.6 and Theorem 2.7. It is useful, however, also to apply Proposition 3.1 because this provides an approximation result which allows to find conditions under which the semigroup is bounded on ℓ^{11} and has a strictly negative essential growth bound [18]. Let

$$\alpha_{jk}^{[n]} = \begin{cases} \alpha_{jj} & \text{if } j = k, \\ \alpha_{jk} & \text{if } j, k < n, j \neq k, \\ 0 & \text{otherwise} \end{cases} \quad (j, k, n \in \mathbb{N}_0).$$

Define $A_n: D_0 \rightarrow \ell^1$ analogously to A_{∞} with $\alpha_{jk}^{[n]}$ replacing α_{jk} , for $n \in \mathbb{N}_0$. Note that in this case the operators $B_n := A_n - A_0$ are bounded (positive) operators.

4.3 Theorem. *The operators A_n generate substochastic C_0 -semigroups S_n on ℓ^1 , and $\|S_n(t)x - S(t)x\| \rightarrow 0$ as $n \rightarrow \infty$ for each $x \in \ell^1$. These semigroups leave ℓ^{11} invariant and induce C_0 -semigroups on ℓ^{11} , and $\|S_n(t)x - S(t)x\|_1 \rightarrow 0$ as $n \rightarrow \infty$ for each $x \in \ell^{11}$. The convergence is monotone increasing, if x is positive.*

In order to illustrate Assumption 4.1, we consider a birth and death process with catastrophes and immigration.

We assume that (α_{jk}) is a Kolmogorov matrix satisfying

$$\alpha_{jk} = 0 \quad \text{for all } j, k \in \mathbb{N}_0 \text{ with } j > k + 1. \quad (4.2)$$

This means that birth rates are such that populations can only increase by one. On the other hand the model allows drastic decreases, including catastrophes wiping out the whole population. This model was also treated in [17; § 4, Example 4]. We indicate a condition implying Assumption 4.1.

4.4 Lemma. *If (4.2) is satisfied then Assumption 4.1 holds if there exists $a > 1$ such that*

$$s := \sup_{k \geq 0} \frac{1}{k+1} (a \alpha_{k+1,k} - \sum_{j=0}^{k-1} (k-j) \alpha_{jk}) < \infty \quad (4.3)$$

Proof. The matrix (α_{jk}) being a Kolmogorov matrix satisfying (4.2) implies

$$\sum_{j=0}^{\infty} j \alpha_{jk} = \alpha_{k+1,k} + \sum_{j=0}^{k-1} (j-k) \alpha_{jk} \quad (k \in \mathbb{N}_0).$$

Because of this equation Assumption 4.1 can be reformulated as

$$(1 + \varepsilon) \alpha_{k+1,k} \leq c(1 + k) + \sum_{j=0}^{k-1} (k-j-\varepsilon) \alpha_{jk} \quad (k \in \mathbb{N}_0).$$

It is not difficult to see that (4.3) implies these inequalities if $(1+\varepsilon)a \leq (1-\varepsilon)$, i.e. $0 < \varepsilon \leq \frac{a-1}{a+1}$, and $c \geq \frac{1-\varepsilon}{a} s$. \square

5 A measure-valued generalization of Kolmogorov's differential equation

Let (Ω, \mathcal{A}) be a measurable space. We consider a measure-valued generalization of Kolmogorov's differential equations,

$$\frac{d}{dt} \mu(t)(\Gamma) = \int_{\Omega} K(\Gamma, x) \mu(t)(dx) - \int_{\Gamma} K(\Omega, x) \mu(t)(dx) \quad (\Gamma \in \mathcal{A}).$$

K is a transition measure kernel: For each $x \in \Omega$, $K(\cdot, x)$ is a non-negative finite measure on (Ω, \mathcal{A}) and for each $\Gamma \in \mathcal{A}$, $K(\Gamma, \cdot)$ is \mathcal{A} -measurable on Ω . The solution $\mu(t)$ takes its values in the Banach space X of signed measures on (Ω, \mathcal{A}) which have bounded variation. If

the values are probability measures, they are associated with the transition probabilities of a Markov jump process [12; 4.2] [14; X.3].

Let $h := K(\Omega, \cdot)$. Then the operator A_0 given by

$$D(A_0) := \left\{ \mu \in X; \int_{\Omega} h(x)|\mu|(dx) < \infty \right\},$$

$$(A_0\mu)(\Gamma) := - \int_{\Gamma} h(x)\mu(dx),$$

generates the C_0 -semigroup S_0 ,

$$(S_0(t)\mu)(\Gamma) = \int_{\Gamma} e^{-th(x)}\mu(dx).$$

The denseness of $D(A_0)$ can be seen as follows: Set $\Omega_n = \{x \in \Omega; h(x) \leq n\}$. Let $\mu \in X$. Define $\mu_n(\Gamma) = \mu(\Gamma \cap \Omega_n)$. Then $\mu_n \in D(A_0)$ and $\mu_n \rightarrow \mu$ as $n \rightarrow \infty$ because Ω is the union of the increasing sequence of sets Ω_n .

We define a positive linear operator $B: D(A_0) \rightarrow X$,

$$(B\mu)(\Gamma) = \int_{\Omega} K(\Gamma, x)\mu(dx).$$

Let $\eta: \Omega \rightarrow [0, \infty)$ be \mathcal{A} -measurable. If we choose X_1 to be the space of signed measures μ with $\|\mu\|_1 = \int_{\Omega} (1 + \eta(x))|\mu|(dx) < \infty$, we obtain the following result from Theorem 2.2, Proposition 2.6 and Theorem 2.7. Notice that the density of X_1 in X follows in the same way as the density of $D(A_0)$.

5.1 Proposition. (a) *There exists a smallest positive C_0 -semigroup S on X whose generator extends $A_0 + B$. S is substochastic.*

(b) *Assume that there exist positive constants c, ε such that*

$$\int_{\Omega} (\eta(y) - \eta(x))K(dy, x) \leq c(1 + \eta(x)) - \varepsilon K(\Omega, x) \quad \text{for all } x \in \Omega.$$

Then S is a stochastic C_0 -semigroup which is generated by the closure of $A_0 + B$, leaves X_1 invariant and induces a C_0 -semigroup on $(X_1, \|\cdot\|_1)$.

We now show that the semigroup S is associated with a Markov transition function. We first note that $Y = \text{BM}(\Omega)$, the space of bounded \mathcal{A} -measurable functions on Ω with the supremum norm, can be identified with a subspace of X^* . The dominated convergence theorem implies that $\text{BM}(\Omega)$ is weakly* sequentially closed. $\text{BM}(\Omega)$ is invariant under $((\lambda - A_0)^{-1})^*$ and $(B(\lambda - A_0)^{-1})^*$, because

$$((\lambda - A_0)^{-1})^* f(x) = \frac{f(x)}{\lambda + h(x)},$$

$$(B(\lambda - A_0)^{-1})^* f(x) = \int_{\Omega} \frac{f(y)}{\lambda + h(x)} K(dy, x)$$

for all $x \in \Omega$, $f \in \text{BM}(\Omega)$. By Proposition 2.4, $\text{BM}(\Omega)$ is invariant under S^* . Set

$$P_t(\Gamma, x) := (S(t)\delta_x)(\Gamma) \quad (\Gamma \in \mathcal{A}), \quad (5.1)$$

where δ_x is the Dirac measure concentrated at x . Then $P_t(\cdot, x)$ is a non-negative measure on \mathcal{A} with values in $[0, 1]$. Since also $P_t(\Gamma, \cdot) = S^*(t)\chi_\Gamma$ and $\text{BM}(\Omega)$ is invariant under $S^*(t)$, $P_t(\Gamma, \cdot) \in \text{BM}(\Omega)$ and

$$(S(t)\mu)(\Gamma) = \int_{\Omega} P_t(\Gamma, x)\mu(dx).$$

The semigroup property of S implies that P satisfies the Chapman-Kolmogorov equations

$$P_{t+s}(\Gamma, x) = \int_{\Omega} P_t(\Gamma, y)P_s(dy, x),$$

i.e., P is a *Markov transition function* [21; Sec. 3.2]. Since S is a C_0 -semigroup, the definition (5.1) shows that for fixed $x \in \Omega$, the function $t \mapsto P_t(\Gamma, x)$ is continuous on $[0, \infty)$, uniformly for $\Gamma \in \mathcal{A}$. If the assumption in Proposition 5.1(b) is satisfied, then $P_t(\cdot, x)$ is a probability measure, and therefore P is a *normal* Markov transition function.

Appendix: A version of the Miyadera perturbation theorem

In this appendix we assume that X is an ordered Banach space with a generating (closed) cone X_+ . We start with an observation that will be needed in the proof of the main result of this section.

A.1 REMARKS. (a) Under the above hypothesis the positive cone X_+ is *non-flat*, i.e., there exists $M \geq 1$ such that for all $x \in X$, $\|x\| \leq 1$, there exist $x_{\pm} \in X_+$ with $\|x_{\pm}\| \leq M$, $x = x_+ - x_-$; cf. [9; Proposition 19.1(d)], [10; p. 265].

This immediately implies that any linear operator $C: X \rightarrow X$ satisfying $c := \sup\{\|Cx\|; x \in X_+, \|x\| \leq 1\} < \infty$ is bounded, $\|C\| \leq 2Mc$.

(b) Let A be the generator of a positive C_0 -semigroup S on X . Then (a) can be strengthened as follows.

Let $C: D(A) \rightarrow X$ be linear, A -bounded, $c := \sup\{\|Cx\|; x \in D(A)_+, \|x\| \leq 1\} < \infty$. Then C uniquely extends to an operator $C \in L(X)$ satisfying $\|C\| \leq 2Mc$, $\sup\{\|Cx\|; x \in X_+, \|x\| \leq 1\} = c$.

In fact, let $x \in D(A)$, $\|x\| \leq 1$. There exist $x_{\pm} \in X_+$, $\|x_{\pm}\| \leq M$, $x = x_+ - x_-$. For λ larger than the type of S one has

$$\begin{aligned} \lambda(\lambda - A)^{-1}x &= \lambda(\lambda - A)^{-1}x_+ - \lambda(\lambda - A)^{-1}x_-, \\ \lambda(\lambda - A)^{-1}x_{\pm} &\in D(A)_+. \end{aligned}$$

Also, $\lambda(\lambda - A)^{-1}x_{\pm} \rightarrow x_{\pm}$ ($\lambda \rightarrow \infty$), and $\lambda(\lambda - A)^{-1}x \rightarrow x$ in the A -graph norm ($\lambda \rightarrow \infty$). Taking $\lambda \rightarrow \infty$ in

$$\|C\lambda(\lambda - A)^{-1}x\| \leq c(\|\lambda(\lambda - A)^{-1}x_+\| + \|\lambda(\lambda - A)^{-1}x_-\|)$$

we obtain $\|Cx\| \leq c(\|x_+\| + \|x_-\|) \leq 2Mc$.

Now $D(A)$ being dense implies that C extends as asserted. Since $D(A)_+$ is dense in X_+ one also obtains the last equality.

A.2 Theorem. Let S_0 be a positive C_0 -semigroup on X , with generator A_0 . Let $B: D(A_0) \rightarrow X$ be positive. Assume that there are constants $0 < \alpha \leq \infty$, $\gamma \in [0, 1)$ such that

$$\int_0^\alpha \|BS_0(t)x\| dt \leq \gamma \|x\| \quad (x \in D(A_0)_+). \quad (\text{A.1})$$

Then $A_0 + B$ is the generator of a positive C_0 -semigroup S .

A.3 REMARKS. (a) If X, Y are ordered Banach spaces such that X_+ is generating (i.e., $X = X_+ - X_+$) and Y_+ is proper (i.e., $Y_+ \cap (-Y_+) = \{0\}$), then any positive linear operator $T: X \rightarrow Y$ is bounded; cf. [1; Appendix].

This implies that, in Theorem A.2, the operator B is A_0 -bounded.

(b) Note that (A.1) only implies

$$\int_0^\alpha \|BS_0(t)x\| dt \leq 2M\gamma \|x\| \quad (x \in D(A_0))$$

(with a proof as in Remark A.1(b)), so the assertion of Theorem A.2 is not a direct consequence of the Miyadera perturbation theorem; cf. [22; Theorem 1], [11; chap. 3, Theorem 3.14]. (For the application of the Miyadera perturbation theorem the constant $2M\gamma$ would have to be < 1 .)

(c) The proof will show that

$$\begin{aligned} & \sup \{ \|S(t)x\|; 0 \leq t < \alpha, x \in X_+, \|x\| \leq 1 \} \\ & \leq \frac{1}{1-\gamma} \sup \{ \|S_0(t)x\|; 0 \leq t < \alpha, x \in X_+, \|x\| \leq 1 \}. \end{aligned}$$

Proof of Theorem A.2. As mentioned above in Remark A.3(a), the operator B is A_0 -bounded.

By induction we define strongly continuous mappings $S_n: [0, \alpha) \rightarrow L(X)$ satisfying

$$\begin{aligned} S_n(t)x &:= \int_0^t S_{n-1}(t-s)BS_0(s)x ds \quad (x \in D(A_0), 0 \leq t < \alpha), \\ \|S_n(t)x\| &\leq \gamma^n \|x\| \quad (x \in X_+, 0 \leq t < \alpha), \end{aligned} \quad (\text{A.2})$$

for all $n \in \mathbb{N}$. Indeed, the linear mappings $S_n(t)$ defined by (A.2) belong to $L(D_{A_0}, X)$, where D_{A_0} denotes $D(A_0)$ provided with the graph norm. The induction hypothesis implies

$$\|S_n(t)x\| \leq \gamma^{n-1} \int_0^t \|BS_0(s)x\| ds \leq \gamma^n \|x\| \quad (x \in D(A_0)_+).$$

Then use Remark A.1(b).

The series $S(t) := \sum_{n=0}^\infty S_n(t)$ is norm convergent, uniformly for $0 \leq t < \alpha$, and as a consequence, S is strongly continuous. The proof that S satisfies the semigroup property on $[0, \alpha)$ and that S can be extended to a C_0 -semigroup is the same as in [22; proof of Theorem 1].

It is easy to see that the generator A of S is an extension of $A_0 + B$ (cf. [22; Lemma 3]). Therefore

$$(\lambda - A_0)^{-1} = (\lambda - A)^{-1}(I - B(\lambda - A_0)^{-1})$$

for large λ . In order to show $D(A) = D(A_0)$ (then $A = A_0 + B$) it therefore is sufficient to show $\text{spr}(B(\lambda - A_0)^{-1}) < 1$ (spr denoting the spectral radius). Let $\gamma' \in (\gamma, 1)$. The estimate

as on [22; p. 168] shows that there exists $\lambda > 0$ such that $\|B(\lambda - A_0)^{-1}x\| \leq \gamma'\|x\|$ for all $x \in D(A_0)_+$. Since $B(\lambda - A_0)^{-1}$ is bounded we obtain $\|B(\lambda - A_0)^{-1}x\| \leq \gamma'\|x\|$ for all $x \in X_+$. Therefore $\|(B(\lambda - A_0)^{-1})^n x\| \leq \gamma'^n \|x\|$ for all $x \in X_+$, $n \in \mathbb{N}$ (it is at this point where the positivity of B is used), and finally $\|(B(\lambda - A_0)^{-1})^n\| \leq 2M\gamma'^n$ ($n \in \mathbb{N}$). These inequalities show $\text{spr}(B(\lambda - A_0)^{-1}) \leq \gamma'$. \square

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