Differentiability of convolutions, integrated semigroups of bounded semi-variation, and the inhomogeneous Cauchy problem

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Inhomogeneous Cauchy Problems

$$u'(t) = Au(t) + f(t), \quad t \in [0,b], \qquad u(0) = 0, \qquad (1)$$

where A a closed linear operator in a Banach space X,

$b \in (0, \infty)$ and $f : [0,b] \to X$.

A function $u : [0,b] \to X$ is called a classical solution of (1) if

- u is continuously differentiable on $[0,b]$,
- $u(t) \in D(A)$ for all $t \in [0,b]$,
- and u satisfies equation (1).

C_0-Semigroups

Travis (1981)

A is the generator of a C_0-semigroup T:

(1) has a classical solution for every continuous $f : [0, b] \to X$ if and only if

the semigroup T is of bounded semi-variation.

Quite restrictive (Travis): implies

- semigroup is analytic
- A is bounded if X is reflexive or an abstract L space.

Where to go from here

smaller classes of inhomogeneities f:

Crandall, Pazy (1969) Hölder continuous.

Webb (1977) continuous and of bounded variation.

Sell, You (2002) sufficient conditions.

Alternative: weaker notion of solution

Integral solutions

If $f \in L^1(0, b, X)$, a function u is an integral solution of (1):

- u is continuous on $[0, b]$,
- $\displaystyle\int_0^t u(s)ds \in D(A)$ for all $t \in [0, b]$,
- $$u(t) = A \int_0^t u(s)ds + \int_0^t f(s)ds, \qquad t \in [0, b].$$

Every classical solution is an integral solution.

Hille-Yosida Operators

There exist $M \geq 1, \omega \in \mathbb{R}$ such that (ω, ∞) is contained in the resolvent set of A and

$$\|(\lambda - A)^{-n}\| \leq M(\lambda - \omega)^{-n}, \quad \lambda > \omega, \; n = 1, 2, \ldots \tag{2}$$

Theorem [Da Prato, Sinestrari, 1987]. Let A be a Hille-Yosida operator, $0 < b < \infty$.

For all $f \in L^1(0, b, X)$, there exist a unique integral solution u of (1).

Estimate

$$\|u(t)\| \leq M \int_0^t e^{\omega(t-s)} \|f(s)\| ds, \qquad 0 \leq t \leq b.$$

Characterization

Proposition. The following are equivalent for a closed linear operator A in a Banach space X.

1. A is Hille-Yosida operator.

2. A is the generator of an integrated semigroup T that is Lipschitz continuous on *some* interval $[0, b]$, $b > 0$.

Arendt (1987), Kellermann&Hieber (1989)

Integrated Semigroups

Focus on integrated semigroups rather than C_0-semigroups.

Arendt, Batty, Hieber, Neubrander (2000)

T is a strongly continuous operator family $T = \{T(t); t \geq 0\}$,

$$T(t)T(r) = \int_0^{t+r} T(s)ds - \int_0^t T(s)ds - \int_0^r T(s)ds, \quad t, r \geq 0,$$
$$T(0) = 0.$$

Example: $T(t) = \displaystyle\int_0^t S(r)dr$, with C_0-semigroup S.

Non-degenerate: $T(t)x = 0$ for all $t > 0$ occurs only for $x = 0$.

Generator

If T is exponentially bounded, definition via Laplace transform,

$$(\lambda - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} T(t) dt$$

for sufficiently large $\lambda > 0$.

Otherwise (Th 1990): if $x, y \in X$,

$$x \in D(A), y = Ax \iff T(t)x - tx = \int_0^t T(s)y\, ds \quad \forall t \geq 0.$$

Integrated solutions

Variation of constants formula, convolution

$$(T * f)(t) = \int_0^t T(t-s)f(s)ds.$$

Proposition. Let A be the generator of an integrated semigroup T, $f \in L^1(0,b,X)$.

Then $v = T * f$ is the unique solution of

$$v(t) = A\int_0^t v(s)ds + \int_0^t (t-s)f(s)ds, \qquad 0 \le t \le b.$$

Can we differentiate?

Semivariation

Hönig (1975). The semi-variation of T on an interval $[a,b]$ is

$$\mathbf{V}_\infty(T;a,b) = \sup \left\| \sum_{j=1}^{k} [T(t_j) - T(t_{j-1})]x_j \right\|,$$

where the supremum is taken over

all partitions $P = \{t_0, \ldots, t_k\}$ of $[a,b]$, $\quad a = t_0 < \cdots < t_k = b$,

and all elements $x_1, \ldots, x_k \in X$, $\|x_j\| \leq 1$, $\quad k \in \mathbb{N}$.

If $\mathbf{V}_\infty(T;a,b) < \infty$, we will say that T is of bounded semi-variation on $[a,b]$.

Semi-p-variation

A larger net (Magal Ruan, 2007):

The semi-p-variation $\mathbf{V}_p(T; a, b)$ is defined by the same formula as before ,

and the supremum is taken over all partitions P (as before)

but $x_1, \ldots, x_k \in X$ with

$$\sum_{j=1}^{k} (t_j - t_{j-1}) \|x_j\|^p \leq 1.$$

Differentiability of convolutions

Theorem. The following hold for a strongly continuous family $T = \{T(t); 0 \le t \le b\}$ of bounded linear operators between Banach spaces X, Y:

(a) $T * f$ is continuously differentiable on $[0, b]$ for all $f \in C(0, b, X)$ if and only if T is of bounded semi-variation on $[0, b]$.

(b) If $1 \le p < \infty$, $T * f$ is continuously differentiable on $[0, b]$ for all $f \in L^p(0, b, X)$ if and only if T is of bounded semi-p-variation on $[0, b]$ and $T(0) = 0$.

In (a), $C(0, b, X)$ can be replaced by $R(0, b, X)$, the space of regulated functions.

Tools

$\Longrightarrow$: Closed graph theorem

$\Longleftarrow$: Stieltjes convolution.

If T is strongly continuous and if $T(0) = 0$ or g is continuous, then $T * g$ is continuously differentiable on $[0, b]$, and

$$(T * g)'(t) = (T \star g)(t) + T(0)g(t).$$

where

$$(T \star g)(t) = \int_0^t T(ds)g(t-s).$$

Estimate

$$\sup_{t\in[0,b]} \|(T \star g)(t)\| \leq \mathbf{V}_p(T; 0, b) \, \|g\|_p.$$

Back to the Cauchy problem

Theorem. Let A be the generator of an integrated semigroup T, $0 < b < \infty$.

(a) If T is of bounded semi-variation on $[0, b]$, then for every $f \in R(0, b, X)$ there exists a unique integral solution u of (1) on $[0, b]$, $u = (T * f)' = T \star f$.

(b) T is of bounded semi-variation on $[0, b]$ if for every $f \in C(0, b, X)$ there exists an integral solution u of (1) on $[0, b]$.

Estimate leading to semilinear theory

$$\sup_{t \in [0,b]} \|(T \star f)(t)\| \leq \mathbf{V}_\infty(T; 0, b) \sup_{t \in [0,b]} \|f(t)\|.$$

Cauchy problem in L^p

Theorem. Let A be the generator of an integrated semigroup T, $1 \le p < \infty$.

Then T is of bounded semi-p-variation on $[0, b]$

if and only if,

for every $f \in L^p(0, b, X)$, there exists an integral solution u of (1) on $[0, b]$.

The solution u is uniquely determined by f, $u = (T * f)' = T \star f$.

Duality characterization in Magal&Ruan (2007).

(q-) variation

The variation of a function $g : [a, b] \to X^*$ is

$$\mathbf{v}(g; a, b) = \sup \sum_{j=1}^{k} \big\| g(t_j) - g(t_{j-1}) \big\|,$$

where the supremum is taken over all partitions $P = \{t_0, \ldots, t_k\}$ with $a = t_0 < \cdots < t_k = b$,

and the q-variation is

$$\mathbf{v}_q(g; a, b) = \sup \left(\sum_{j=1}^{k} \frac{\big\| g(t_j) - g(t_{j-1}) \big\|^q}{(t_j - t_{j-1})^{q-1}} \right)^{1/q}.$$

The duality connection

T^* is called to be of bounded strong (q-)variation if $T^*(\cdot)y^*$ is of bounded (q-)variation for each $y^* \in Y^*$.

Operator families of bounded strong (p-) variation are studied by Th&Voßeler (2002) and Voßeler (2000).

Proposition. T is of bounded semi-variation if and only if T^* is of bounded strong variation.

If $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

T is of bounded semi-p-variation

if and only if

T^* is of bounded strong q-variation.

More on semi-p-variation

Lemma. (a) If T is of bounded semi-p-variation, then it is Hölder continuous with exponent $1/p$ and

$$\|T(t) - T(s)\| \leq \mathbf{V}_p(T; a, b)|t - s|^{1/p}.$$

(b) T is of bounded semi-1-variation if and only if it is Lipschitz continuous, and

$$\mathbf{V}_1(T; a, b) = \sup_{a \leq s < t \leq b} \frac{\|T(t) - T(s)\|}{t - s}.$$

Characterization of Hille-Yosida Ops.

Theorem. The following are equivalent for a closed linear operator A in a Banach space X.

1. A is a Hille-Yosida operator.

2. (i) For each $x \in X$, there exists a unique integral solution of

$$u' = Au + x \text{ on } \mathbb{R}_+, \quad u(0) = 0,$$

which is exponentially bounded.

(ii) There exists some $b \in (0, \infty)$ such that, for each $f \in L^1(0, b, X)$, there exists a unique integral solution u of (1) on $[0, b]$.

Example

Let $p > 1$, $\quad 0 < \alpha < \frac{p-1}{p}$, $\quad X = L^p[0,1]$,

$$(Af)(x) = -f'(x) + \frac{\alpha}{x} f(x).$$

$D(A)$: absolutely continuous functions f on $[0,1]$ with $f(0) = 0$ and $f' \in L^p[0,1]$,

Arendt (1987): A generates the integrated semigroup

$$[T(t)f](x) = \int_0^t x^\alpha (x-s)^{-\alpha} f(x-s) H(x-s)\, ds, \qquad x \in [0,1].$$

H is the Heaviside function. T is not a C_0-semigroup.
Since $D(A)$ is dense, A is not a Hille-Yosida operator.

Example cont.

The dual family on $L^q[0,1]$, $\frac{1}{q}+\frac{1}{p}=1$, is given by

$$[T^*(t)g](x) = \int_0^t (x+s)^\alpha x^{-\alpha} g(x+s) H(1-x-s)\,ds,$$

T^* is the integrated semigroup generated by A^*.

T^* is of locally bounded strong q-variation.

T is of locally bounded semi-p-variation.

Notice: X is an ordered Banach space,
T and T^* are increasing,
A and A^* have positive resolvents.

Age-structure

Find $u(t,a) \in X$, t time, a age,

X Banach space, additional structure (space e.g.)

$$\begin{aligned} u_t + u_a = & \; B(a)u + g(t,a), && t > 0, a > 0, \\ u(t,0) = & \; h(t), && t > 0, \\ u(0,a) = & \; u_0(a), && a > 0. \end{aligned}$$

$$u(t,\cdot) \in L^p(\mathbb{R}_+, X), \qquad 1 \leq p < \infty.$$

$B(a)$ closed linear operators associated with an evolutionary system $\{U(a,s); a \geq s\}$.

A larger space

$$\mathcal{X} = X \times L^p(\mathbb{R}_+, X), \qquad \mathcal{X}_0 = \{0\} \times L^p(\mathbb{R}_+, X).$$

$$\mathcal{B}(0,\phi) = \Big(-\phi(0),\ -\phi' + B(\cdot)\phi\Big).$$

Set $\quad v(t) = (0, u(t,\cdot))$, $\qquad$ Cauchy problem

$$v'(t) = \mathcal{B}v(t) + (h(t), g(t,\cdot)).$$

$\mathcal{B}$ is a Hille-Yosida operator if $p = 1$ (Th 1989),

and the generator of an integrated semigroup of locally bounded semi-p-variation if $p > 1$ (Magal&Ruan 2007).

Evolution semigroups

Let $U = \{U(t,s); 0 \le s \le t < \infty\}$ be the forward evolutionary system associated with $B(t)$.

We define the associated (*Howland*) evolution semigroup on $Y = L^p(\mathbb{R}_+, X)$, $1 \le p < \infty$, by

$$[S(t)\phi](a) = \left\{ \begin{array}{cc} U(a, a-t)\phi(a-t); & 0 \le t \le a \\ 0; & 0 \le a < t \end{array} \right\}, \ \phi \in Y.$$

Integrated semigroup $\quad T(t)(x,\phi) = (0, \psi(t)),$

$$\psi(t)(a) = H(t-a)U(a,0)x + \int_0^t [S(r)\phi](a)dt.$$

Perturbation Theory

Generalization of a result by Da Prato and Grisvard (1975).

Theorem. Let $S = \{S(t); t \geq 0\}$ be a C_0-SG, generator A,

$T = \{T(t)\}$ an ISG, generator B.

Assume that T is of locally bounded semi-variation and that T and S commute, i.e. $T(t)S(r) = S(r)T(t)$ for all $t, r \geq 0$. Then

$$\tilde{T}(t)x = \int_0^t T(dr)S(r)x$$

defines an ISG of locally bounded semi-variation whose generator extends $A + B$ (with domain $D(A) \cap D(B)$). $\tilde{T}$ commutes with both T and S.

commuting families

If $0 < b < \infty$ and T is of bounded semi-p-variation on $[0, b]$, so is $\tilde{T}$ and

$$\mathbf{V}_p(\tilde{T}; 0, b) \leq m\mathbf{V}_p(T; 0, b)$$

$$m = \sup_{0 \leq t \leq b} \|S(t)\|.$$

Da Prato and Grisvard (1975): $p = 1$

Bounded perturbations

Theorem. Let A be the generator of an ISG T which is of locally bounded semi-(p-)variation and the linear operator $B : D(A) \to X$ satisfy

$$\|B\| = \sup\{\|Bx\|; x \in D(A), \|x\| \leq 1\} < \infty.$$

If T is of locally bounded semi-variation, assume in addition that $\|B\|\mathbf{V}_\infty(T, 0, b) < 1$ for some $b > 0$.
Then $A + B$ generates an ISG V of locally bounded semi-(p-)variation which solves the equations

$$V(t)x - T(t)x = \int_0^t T(ds)\bar{B}V(t-s)x = \int_0^t V(ds)\bar{B}T(t-s)x,$$

where $\overline{B}$ is the extension of B to $\overline{D(A)}$.

Summary

(CP) $$u' = Au + f(t) \quad \text{on } [0,b], \qquad u(0) = 0.$$

- (CP) has integral solutions for all continuous (regulated) f if and only if A generates an ISG of bounded semi-variation.
- (CP) has integral solutions for all $f \in L^p$ if and only if A generates an ISG of bounded semi-p-variation.
- the generator A of an ISG is a Hille-Yosida operator if and only if there is some $b > 0$ such that (CP) has integral solutions for all $f \in L^1$.

Perturbations

- ISGs of locally bounded semi-(p-)variation are preserved under bounded additive perturbations of their generators.
- The commutative sum $A + B$ of generators of a C_0-semigroup and of an ISG of locally bounded semi-(p-)variation generates an ISG of the same type.

Examples

- ISGs of bounded semi-p-variation occur naturally in age-structured population models (cf. Magal Ruan, 2007)

  where, at any time, the age-distribution of the population is a vector-valued function in $L^p(\mathbb{R}_+, X)$.

- Natural examples of resolvent positive operators generate ISGs of bounded semi-(p-)variation.