Global compact attractors and their tripartition under persistence

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GCOE, September 2009

Overview

Dynamical systems, semiflows

dynamics of populations (human, animal, plant)

populations persistence (survival of the population)

(global) compact attractors

How do persistence and compact attractors interact?

The existence of a global compact attractor facilitates persistence.

Persistence divides the global compact attractor into three parts:
extinction attractor, persistence attractor, transient set.

Dynamical Systems and Population Persistence
American Mathematical Society, Graduate Series, 2011/12

Semiflows and their state spaces

The temporal development of a natural or artificial system can conveniently be modeled by a semiflow.

A semiflow consists of a state space, X, a time-set, J, and a map, Φ.

The state space X comprehends all possible states of the system:

the amounts or densities of the system parts;
if structure, their structural distribution.

epidemiological system: the amounts or densities of susceptible and infective and possibly exposed and removed individuals.

For spatial spread, spatial distributions
age-structure: age-distributions

Time set

time-set J: $\mathbb{R}_+ = [0, \infty)$ or $\mathbb{Z}_+ = \mathbb{N} \cup \{0\} = \{0, 1, \dots\}$.

Definition

A subset $J \subseteq [0, \infty)$ is called a time-set if it has the following properties:

1. $0 \in J$ and $1 \in J$.
2. If $s, t \in J$, then $s + t \in J$.
3. If $s, t \in J$, and $s < t$, then $t - s \in J$.

J is a time-set iff $\hat{J} = J \cup (-J)$ is a subgroup of $(\mathbb{R}, +)$ containing $\mathbb{Z}$.

If J is closed, then $J = [0, \infty)$ or $J = \{m/n; m \in \mathbb{Z}_+\}$ where $n \in \mathbb{N}$.
Depending on the model, the time unit can be a year, month, or day.

Semiflow map

$$\Phi : J \times X \to X.$$

Often Φ itself is called the semiflow. If $x \in X$ is the initial state of the system (at time 0), then $\Phi(t, x)$ is the state at time t.

$$\Phi(0, x) = x, \qquad x \in X.$$

Further, semiflows are characterized by the semiflow property:

$$\Phi(t + r, x) = \Phi(t, \Phi(r, x)), \qquad r, t \in J, \quad x \in X.$$

Write $\Phi_t(x) = \Phi(t, x)$, $\qquad \Phi_t : X \to X$, $\qquad \Phi_t \circ \Phi_r = \Phi_{t+r}$.

An endemic model with vaccination and return to susceptibility

We consider a population of constant size,
proportion of susceptible individuals at time t, $S(t)$,
of infected individuals, $I(t)$, of removed individuals, $R(t)$,
and of vaccinated individuals, $V(t)$,

$$1 = S(t) + I(t) + R(t) + V(t).$$

$$\begin{aligned}
S' =& \mu - \mu S - \kappa S - \sigma S I + \theta R + \eta V, \\
I' =& \sigma S I - (\gamma + \mu) I, \\
R' =& \gamma I - (\theta + \mu) R, \\
V' =& \kappa S - (\eta + \mu) V.
\end{aligned}$$

The solution semiflow

$$
\begin{aligned}
I' =& \sigma(1 - I - R - V)I - (\gamma + \mu)I, \\
R' =& \gamma I - (\theta + \mu)R, \\
V' =& \kappa(1 - I - R) - (\kappa + \eta + \mu)V.
\end{aligned}
\tag{1}
$$

The solutions to this system induce a continuous semiflow on

$$
X = \{(I, R, V) \in \mathbb{R}^3_+;\ I + R + V \leq 1\}. \tag{2}
$$

Solution semiflow

$$
\Phi(t, x_0) = x(t) = (I(t), R(t), V(t)), \qquad x(0) = x_0.
$$

Persistence

Does the dynamical system persist (remains safely away from extinction) as a whole or at least in parts (which parts?).

mathematically formulated by using a persistence function

$$\rho : X \to \mathbb{R}_+. \tag{3}$$

For $x \in X$, $\rho(x)$ is the abundance of the part of the system that is of particular interest.

In our example, host persistence is automatic.
For disease persistence, we choose

$$\rho(I, R, V) = I.$$

We could also choose $\rho(I, R, V) = I + R$.

The semiflow Φ is uniformly ρ-persistent if there exists some $\epsilon > 0$ s.t.

$$\liminf_{t\to\infty} \rho(\Phi(t,x)) \geq \epsilon \quad \text{whenever } x \in X, \rho(x) > 0. \tag{4}$$

Φ uniformly weakly ρ-persistent if (4) holds with $\limsup$ replacing $\liminf$.

OVERALL ASSUMPTIONS

- X is a metric space.
- ρ is continuous
- $\rho \circ \Phi$ is continuous.

Lotka-Volterra predator-prey model:

$$P' = \xi P - PQ, \qquad Q' = PQ - \mu Q$$

is uniformly weakly, but not uniformly persistent.

Persistence with a compact attracting set

Theorem

Let X be a metric space. Assume there exists a compact set A s.t.

- $\Phi(t,x) \to A$ as $t \to \infty$ for every $x \in X$ with $\rho(x) > 0$
- there are no $y \in A$, $s,t \in J$:

$$\rho(y) > 0, \quad \rho(\Phi(s,y)) = 0, \quad \text{and } \rho(\Phi(t+s,y)) > 0.$$

Then Φ is uniformly persistent if it is uniformly weakly persistent.

$\Phi(t,x) \to A$: for every open set $U \supseteq A$ there exists some $r \in J$ s.t. $\Phi(t,x) \in U$ for all $t \in J, t \geq r$.

Hale, Waltman (1989)

Compact attractors

Definition

Let J be a time-set and $\Phi : J \times X \to X$ be a semiflow.

- A set $K \subseteq X$ is said to attract a set $M \subseteq X$, if $K \neq \emptyset$ and $\Phi_t(M) \to K$ as $t \to \infty$.

  We also say that M is attracted by K.
- K is called an attractor of M, if K is invariant and attracts M. In this situation, we also say that M has the attractor K.

$\Phi_t(M) \to K$: for every open set U, $K \subseteq U \subseteq X$, there exists some $r \in J$ s.t. $\Phi_t(M) \subseteq U$ for all $t \in J$, $t \geq r$.

K forward invariant: $\Phi_t(K) \subseteq K$ for all $t \in J$,
K backward invariant: $\Phi_t(K) \supseteq K$ for all $t \in J$,
K invariant: $\Phi_t(K) = K$ for all $t \in J$.

ω-limit set

Definition

The ω-*limit set* of a subset M of X is defined as

$$\omega(M) = \bigcap_{t \in J} \overline{\Phi\big(J_t \times M\big)}, \qquad J_t = J \cap [t, \infty).$$

Obviously, $\omega(M)$ is a (possibly empty) closed set.

Alternative characterization.

Lemma

An element x in X satisfies $x \in \omega(M)$ if and only if there are sequences (t_j) in J, $t_j \to \infty$ as $j \to \infty$, and (x_j) in M such that $\Phi(t_j, x_j) \to x$ as $j \to \infty$.

A result without continuity

Let J be a time-set and $\Phi : J \times X \to X$ a map, $\emptyset \neq M \subseteq X$.

Definition (Sell, You, 2002)

Φ is called asymptotically compact on M, if, for any sequences (t_i) in J, $t_i \to \infty$ as $i \to \infty$, and (x_i) in M, $(\Phi(t_i, x_i))$ has a convergent subsequence.

Theorem

Equivalent:

1. Φ is asymptotically compact on M.
2. M is attracted by a non-empty compact set $K \subseteq X$.
3. $\omega(M)$ is non-empty, compact, and attracts M.

If one and then all of these three statements hold, $\omega(M) \subseteq K$ for every compact $K \subseteq X$ that attracts M.

Compact attractors of individual sets

Φ is called state-continuous if each map Φ_t is continuous.

Theorem

Let J be a time-set and $\Phi : J \times X \to X$ a state-continuous semiflow, $\emptyset \neq M \subseteq X$.

Then every backward invariant subset of $\overline{\Phi(J \times M)}$ is contained in $\omega(M)$.

If M is attracted by a compact set or, equivalently, Φ is asymptotically compact on M, then M has a compact attractor, namely $\omega(M)$.

$\omega(M)$ is the unique compact attractor of M contained in $\overline{\Phi(J \times M)}$.

If C is a subset of X and attracts $\omega(M)$, then C attracts M.

Compact attractor of classes of sets

We assume that Φ is state-continuous.

Definition

Let $\mathcal{C}$ denote a class of subsets of X.
A non-empty, compact, invariant set $K \subseteq X$ is called the compact attractor of $\mathcal{C}$ if K attracts all sets in $\mathcal{C}$.

Theorem

The following are equivalent for a class $\mathcal{C}$ of subsets of X.

(a) There exists a compact attractor of $\mathcal{C}$.

(b) Φ is asymptotically compact on every set $M \in \mathcal{C}$ and $\bigcup_{M \in \mathcal{C}} \omega(C)$ has compact closure in X.

(c) There is a compact set in X that attracts every set $M \in \mathcal{C}$.

If (a), (b) or (c) and then all of them hold, the closure of $\bigcup_{M \in \mathcal{C}} \omega(C)$ is the smallest compact attractor of $\mathcal{C}$.

"Global" compact attractor

We do not use the notion of a global attractor because there is no agreement in the literature about this term.

Definition

Instead we use the following terminology for a non-empty compact invariant set A:

If $\mathcal{C}$ is the class of singleton sets in X and A attracts $\mathcal{C}$, A is called a compact attractor of points.

If $\mathcal{C}$ is the class of bounded (compact) sets in X and A attracts $\mathcal{C}$, A is called an (actually the) compact attractor of bounded (compact) sets.

A is a (the) compact attractor of neighborhoods of compact sets if every compact set in X has a neighborhood that is attracted by A.

“Global” compact attractor in the literature

The term global compact attractor has been used in various ways in the literature:

compact attractor of points: Ladyzhenskaya, 1991; Matano, Nakamura, 1997

compact attractor of neighborhoods of compact sets:
Sell, You, 2002; Magal, Zhao, 2005

compact attractor of bounded sets:
Hale 1989; Diekmann, van Gils, Verduyn Lunel, Walter, 1995

exponential attractor (of points)
Eden, Foias, Nicolaenko, Temam (1994)
Osaki, Tsujikawa, Yagi, Mimura (2002)

Applicable concepts for global compact attractors

Definition (Hale, 1989)

Let $\Phi : J \times X \to X$ be a state-continuous semiflow.

- Φ is called point-dissipative (or *ultimately bounded*) if there exists a bounded subset B of X which attracts all points in X.
- Φ is called asymptotically smooth if Φ is asymptotically compact on every forward invariant bounded closed set.
- Φ is called eventually bounded on a set $M \subseteq X$ if $\Phi(J_r \times M)$, $J_r = J \cap [r, \infty)$, is bounded for some $r \in J$.

If X is a closed subset of $\mathbb{R}^n$, Φ is asymptotically smooth.

More generally, if Φ_r is compact on the metric space X for some $r \in J$, Φ is asymptotically smooth.

Applicable conditions for global compact attractors

Theorem

Let $\Phi : J \times X \to X$ be a state-continuous semiflow. Assume that Φ is point-dissipative and asymptotically smooth. Then there exists a compact attractor of points, namely the closure $\Omega(X)$ of $\bigcup_{x \in X} \omega(x)$.

Next result has been inspired by [Magal, Zhao, 2005]. Also [Hale, 1989].

Theorem

The following are equivalent for a state-continuous semiflow Φ.

(a) Φ is point-dissipative, asymptotically smooth, and eventually bounded on every compact subset K of X.

(b) There exists a compact attractor A of neighborhoods of compact sets in X; A attracts every subset of X on which Φ is eventually bounded.

Attractors of bounded sets

Theorem (Hale, 1989)

The following statements are equivalent:

1. There exists a compact attractor of bounded sets (which is unique and contains every bounded backward invariant set).
2. Φ is point-dissipative, asymptotically smooth, and eventually bounded on every bounded set in X.

There are attractors of neighborhoods of compact sets that are not attractors of bounded sets.

infinite-dimensional examples: Magal, Zhao (2005)
planar example: endemic model

$$S' = \beta S + q\beta I - \mu S - SI, \qquad I' = SI - (\mu + \alpha)I.$$

$\beta > \mu > 0, \quad 0 < q\beta < \mu + \alpha;$ state space: $S > 0$, $I > 0$.

Attractors and stability

Theorem

Let A be a compact attractor of points in X and A be stable. Then A is the compact attractor of neighborhoods of compact sets.

Assume Φ be state-continuous, uniformly in finite time:

$\Phi_t(x)$ is continuous in x uniformly for t in bounded subsets of J.

Theorem

Let A be the compact attractor of compact sets in X.
Then A is the compact attractor of neighborhoods of compact sets in X.
Further A is stable.

Open Problem: Is asymptotic smoothness necessary for existence of a compact attractor of compact sets?

Global attractors are connected

Theorem

Let Φ be state-continuous.

(a) If K is the compact attractor of a connected set B, $K \subseteq B \subseteq X$, then K is connected.

(b) Let X be the closed convex subset of a Banach space. If $K \subseteq X$ is the compact attractor of compact sets in X, then K is connected.

Proof.

(b) The closed convex hull of K is compact by a theorem by Mazur and thus attracted by K. □

Total trajectories

Recall

$$\hat{J} = J \cup (-J).$$

Let Φ be a semiflow on X.

Definition

$\phi : \hat{J} \to X$ is a total Φ-trajectory if

$$\Phi(t, \phi(s)) = \phi(t + s), \qquad t \in J, s \in \hat{J}.$$

In ODEs, a total trajectory is a solution that exists for all times.

Theorem

$A \subseteq X$ is invariant if and only if for every $x \in A$ there is a total Φ-trajectory $\phi : \hat{J} \to A$ with $\phi(0) = x$.

Let $J = \mathbb{R}_+$ or $J = \mathbb{Z}_+ = \{0\} \cup \mathbb{N}$. Assume that the semiflow $\Phi : J \times X \to X$ is state-continuous, $\rho : X \to \mathbb{R}_+$ is continuous and $\rho \circ \Phi$ is continuous.

(H0) Φ has a compact attractor A which attracts all points in X.

(H1) There exists no total trajectory ϕ with range in A such that $\rho(\phi(0)) = 0$ and $\rho(\phi(-r)) > 0$ and $\rho(\phi(t)) > 0$ for some $r, t \in J$.

The following set is closed and forward invariant (though possibly empty),

$$X_0 = \{x \in X;\ \forall t \in J:\ \rho(\Phi(t,x)) = 0\}.$$

Recall that Φ is uniformly weakly ρ-persistent, if there exists $\eta > 0$ s.t.

$$\limsup_{t\to\infty} \rho(\Phi_t(x)) > \eta, \qquad \text{whenever } \rho(x) > 0,$$

and is uniformly ρ-persistent if we can replace $\limsup$ by $\liminf$ above.

An attractor of points facilitates persistence

The following result says roughly that uniform weak persistence plus a compact attractor of points implies uniform strong ρ-persistence.

Theorem

If $X_0 = \emptyset$, then $\rho(x) > 0$ for all $x \in A$ and
there exists some $\eta > 0$ such that $\liminf_{t \to \infty} \rho(\Phi(t, x)) \geq \eta$ for all $x \in X$.

Theorem

If $X_0 \neq \emptyset$ and Φ is uniformly weakly ρ-persistent, then Φ is uniformly ρ-persistent.

$J = \mathbb{Z}_+$ or $J = \mathbb{R}_+$,
the semiflow $\Phi : J \times X \to X$ is state-continuous, uniformly in finite time.

Assume that Φ has a compact attractor, A, of compact sets in X.

A is the compact attractor of neighborhoods of compact sets in X.

Definition

$Y \subseteq X$ is called uniformly ρ-positive if $\inf \rho(Y) > 0.$

The semiflow Φ is eventually uniformly ρ-positive on $D \subseteq X$
if $\Phi(J_r \times D)$ is uniformly ρ-positive for some $r \in J$
where $J_r = \{t \in J : t \geq r\}$.

The partition theorem

Assume that $X_0 \neq \emptyset$, Φ is uniformly weakly ρ-persistent, and

(H1) there exists no total Φ-trajectory ϕ with range in A such that $\rho(\phi(-r)) > 0$, $\rho(\phi(0)) = 0$, and $\rho(\phi(s)) > 0$ with $r, s \in J$,

Then the attractor A is the disjoint union

$$A = A_0 \cup C \cup A_1$$

of three invariant sets A_0, C, and A_1. A_0 and A_1 are compact and

(a) $A_0 = A \cap X_0$ is the compact attractor of neighborhoods of compact subsets in X_0.

A_0 attracts every subset of X_0 that is attracted by A.

(b) A_1 is uniformly ρ-positive and is the compact attractor of neighborhoods of compact sets in $X \setminus X_0$.

A_1 is stable.

A_1 also attracts all sets that are attracted by A and on which Φ is eventually uniformly ρ-positive.

(c) If $x \in X \setminus A_1$ and ϕ is a total Φ-trajectory through x with pre-compact range, then $\phi(t) \to A_0$ as $t \to -\infty$.

If $x \in X \setminus A_0$ and ϕ is total Φ-trajectory through x with pre-compact range, then $\phi(t) \to A_1$ as $t \to \infty$.

C consists of those points $x \in A$ through which there exists a total trajectory ϕ with $\phi(-t) \to A_0$ and $\phi(t) \to A_1$ as $t \to \infty$.

Connectedness of the persistence attractor

A_1 persistence attractor of Φ, A_0 the extinction attractor of Φ.

Existence of persistence attractor: Hale&Waltman 1990, Zhao 2003, Magal&Zhao 2005.

Theorem

Let X be the closed convex subset of a Banach space. Let the assumptions of the Partition Theorem be satisfied and the persistence function ρ be concave. Then the persistence attractor, A_1, is connected.

Proof.

If ρ is concave, the closed convex hull of A_1 is uniformly ρ-positive (and compact) and thus attracted by A_1. □

Existence of non-trivial fixed points via persistence

Assumptions:

$J = \mathbb{Z}_+$ or $J = \mathbb{R}_+$ and Φ is continuous.
X is the closed convex subset of a Banach space,
$\rho : X \to \mathbb{R}_+$ is continuous and concave.

Φ is uniformly weakly ρ-persistent,
has a compact attractor of compact sets in X.

Φ_t is compact (or condensing) for $\begin{cases} t = 1 & \text{if } J = \mathbb{Z}_+ \\ t \in (0, 1] & \text{if } J = \mathbb{R}_+ \end{cases}$.

(H1) there exists no total Φ-trajectory with pre-compact range such that $\rho(\phi(-r)) > 0$, $\rho(\phi(0)) = 0$, and $\rho(\phi(s)) > 0$ with $r, s \in J$.

Conclusion

Then there exists $x^\diamond \in X$ with $\rho(x^\diamond) > 0$ and $\Phi(t, x^\diamond) = x^\diamond$ for all $j \in J$.

[Similar results by Zhao (2003) and Magal and Zhao (2005).]

Back to the endemic model with vaccination

$$
\begin{aligned}
I' =& \sigma(1 - I - R - V)I - (\gamma + \mu)I, \\
R' =& \gamma I - (\theta + \mu)R, \\
V' =& \kappa(1 - I - R) - (\kappa + \eta + \mu)V.
\end{aligned}
$$

state space $\quad X = \{(I, R, V) \in \mathbb{R}^3_+;\ I + R + V \leq 1\}.$

X is compact, so solution semiflow Φ has a compact attractor of bounded sets in X (including X itself).

disease persistence $\quad \rho(I, R, V) = I.$

$$X_0 = \{(0, R, V);\ R \geq 0,\ V \geq 0,\ V + R \leq 1\}.$$

X_0 has a compact attractor of bounded sets, A_0. Since X_0 invariant (not only forward invariant), $A_0 = A \cap X_0$.

Disease-free dynamics

Consider bounded everywhere defined solutions of

$$\begin{aligned} I \equiv & 0, \\ R' = & -(\theta + \mu)R, \\ V' = & \kappa(1 - R) - (\kappa + \eta + \mu)V. \end{aligned}$$

Then $R \equiv 0$ and

$$V \equiv \frac{\kappa}{\kappa + \eta + \mu} =: V^\diamond.$$

Theorem

The disease free equilibrium $(0, 0, V^\diamond)$ attracts X_0.

Disease persistence

$$V' \leq \kappa - (\kappa + \eta + \mu)V.$$

$$V^\infty \leq V^\diamond, \qquad R^\infty \leq \frac{I^\infty}{\theta + \mu}.$$

$$\liminf_{t\to\infty} \frac{I'(t)}{I(t)} \geq \sigma(1 - I^\infty - R^\infty - V^\diamond) - (\gamma + \mu).$$

Theorem

Let

$$1 < \frac{\sigma(1-V^\diamond)}{\gamma+\mu} =: \mathcal{R}. \qquad \text{(net replacement ratio)}$$

Then the disease is uniformly persistent: there exists some $\epsilon > 0$ such that

$$\liminf_{t\to\infty} I(t) \geq \epsilon$$

for all solutions with $I(0) > 0$.

If $\mathcal{R} > 1$, there is a disease persistence attractor.
If $\mathcal{R} \le 1$, there is still the overall attractor.

To describe them, consider everywhere defined bounded solutions.
Transform to an integral equation,

$$I'(t) = I(t)\Big(\tilde{\sigma} - \int_0^\infty I(t-s)m(ds)\Big), \qquad t \in \mathbb{R},$$

with

$$\tilde{\sigma} = \sigma(1 - V^\diamond) - (\gamma + \mu) = (\mathcal{R} - 1)(\gamma + \mu).$$

Let m be a signed Borel measure on $\mathbb{R}_+$, $m(\{0\}) \ge 0$, whose variation $|m|$ satisfies

$$\int_{\mathbb{R}_+} (1+s)|m|(ds) < \infty$$

and whose Fourier transform satisfies

$$\inf_{s \ge 0} \Re \int_{\mathbb{R}_+} e^{-ist} m(dt) > 0.$$

A result for integro-differential equations

Theorem

If $\tilde{\sigma} \le 0$, every bounded non-negative solution is identically 0.
If $\tilde{\sigma} > 0$, every positive solution that is bounded and bounded away from 0 is constant.

Proof: Adapt Fourier transform techniques by London (1983) to everywhere defined solutions.

For $\tilde{\sigma} > 0$ another transformation is required. Set $I^* = \dfrac{\tilde{\sigma}}{m(\mathbb{R}_+)}$,

$$\frac{I'(t)}{I(t)} = \int_0^\infty (I^* - I(t-s))m(ds).$$

Set $u(t) = \ln \dfrac{I(t)}{I^*}$,

$$-u'(t) = \int_0^\infty g(u(t-s))m(ds), \qquad g(u) = I^*(e^u - 1).$$

Global stability of equilibria in the endemic model

Theorem

If $\mathcal{R} \le 1$, then the disease-free equilibrium attracts the whole state space X and is stable.

If $\mathcal{R} > 1$, there is a unique endemic equilibrium. It attracts every subset $X \cap \{I \ge \epsilon\}$ and is stable.

Conclusion

A global compact attractor facilitates persistence.

Persistence creates a tripartition of the global compact attractor.

This is a stepping-stone for proving global asymptotic stability (in a uniform sense) of equilibria.