Semiflows Generated by Lipschitz Perturbations of Non-densely Defined Operators

II. Examples

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In part I of this paper (i.e. sections 1 to 4 of [T2]) we have studied abstract Cauchy problems

$$(\diamondsuit) \qquad \frac{d}{dt}u(t) = Au(t) + F(t)u(t),\ t > t_0 \geq 0; \quad u(t_0) = x_0,$$

with a non-densely defined operator A on a Banach space X and Lipschitz perturbations $F(t) : C_0 \to X$. Here $C_0 = C \cap \overline{D(A)}$ with a closed convex subset C of X. Under suitable subtangential conditions we have found integral solutions u to $(\diamondsuit)$ with values in C_0 provided that $x_0 \in C_0$. More precisely we have found a continuous function $u : [t_0, \infty) \to C_0$ satisfying

$$(\heartsuit) \qquad u(t) = x_0 + A\int_{t_0}^{t} u(s)ds + \int_{t_0}^{t} F(s)u(s)ds,\ t \geq t_0.$$

See section 2. In section 3 and 4 we have demonstrated that this framework is specific enough to study the regularity of the solutions and the properties of the dynamical system generated by the solution flow. Now, in part II, we illustrate that this concept is general enough to cover examples like functional differential equations (section 5), abstract functional differential equations (section 6), age-dependent population dynamics (section 7), age-structured functional differential equations (section 8), and abstract semilinear boundary value evolution problems (section 9).

5. Functional differential equations

We illustrate how nonlinear functional differential equations in $\mathbf{R}^n$ can be written as a perturbation of a non-densely defined generator:

$$(5.1) \qquad \begin{gathered} \frac{d}{dt}x(t) = f(t, x_t);\ t > t_0 \geq 0; \\ x(t_0 + r) = \phi(r);\ -\tau \leq r \leq 0 \end{gathered}$$

with a given continuous function ϕ.

Recall the convention

$$x_t(r) = x(t + r);\ -\tau \leq r \leq 0.$$

Let $\tilde{C}$ be a closed convex set in $\mathbf{R}^n$ and set

$$C_0 = C([-\tau, 0], \tilde{C}).$$

Let

$$f : [t_0, \infty) \times C_0 \to \mathbf{R}^n$$

satisfy the following assumptions:

(i) For any $v \in C_0$, $f(t, v)$ is a continuous function of t.

(ii) For any $t \geq 0, v \in C_0$ there exist $\delta, \Lambda > 0$ such that

$$\|f(s, y) - f(s, z)\| \leq \Lambda \|y - z\|,$$

if $t \leq s \leq t + \delta,\ y, z \in C_0,\ \|y - v\|, \|z - v\| \leq \delta.$

(iii) For any $\rho > 0$ there exists some $c > 0$ such that

$$\|f(t, v)\| \leq c(1 + \|v\|)$$

if $0 \leq t \leq \rho, v \in C_0$.

(iv) For $t \geq 0, v \in C_0$,

$$\frac{1}{h} dist(v(0) + hf(t, v); \tilde{C}) \to 0 \quad as \quad h \downarrow 0.$$

We now note that, at least formally,

$$(\partial_t - \partial_r) x_t(r) = 0;\ \tau < r < 0;$$

$$x_t(0) = x(t).$$

Setting $u(t) = (x(t), x_t)$ we can write the (5.1) in the form ($\diamondsuit$) in section 2 with

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \frac{d}{dr} \end{pmatrix}$$

and

$$F(t)(x, v) = (f(t, v), 0).$$

We choose

$$X = \mathbf{R}^n \times C([-\tau, 0], \mathbf{R}^n),\ X_0 = \{(v(0), v), v \in C([-\tau, 0], \mathbf{R}^n)\},$$

$$C = \{(v(0), w); v, w \in C_0\} = \tilde{C} \times C_0.$$

Note that C_0 can be identified with $C \cap X_0$ and X_0 can be identified with $C([-\tau, 0], \mathbf{R}^n)$. Further

$$D(A) = \{(v(0), v); v \in C^1([-\tau, 0], \mathbf{R}^n)\}.$$

Note that $\overline{D(A)} = X_0$.

The assumptions 2.1 and 2.2 are easily checked as far as F is concerned. Let us consider A.

In order to show assumption 2.1 a) we must solve

$$(\lambda - A)(x, v) = (\tilde{x}, \tilde{v}) \in X.$$

This means

$$\lambda x = \tilde{x},\ \lambda v(r) - v'(r) = \tilde{v}(r),\ v(0) = x.$$

This problem is uniquely solved by

$$x = \frac{1}{\lambda}\tilde{x}, \quad v(r) = \frac{1}{\lambda}\tilde{x}e^{\lambda r} + \int_r^0 e^{\lambda(r-s)}\tilde{v}(s)ds,\ -\tau \leq r \leq 0.$$

Note that $\lambda v(r) \in \tilde{C}$, if $\tilde{v} \in C_0$, because it can then be approximated by a convex combination of elements in $\tilde{C}$. Further $\lambda x = \tilde{x} \in \tilde{C}$ if $\tilde{x} \in \tilde{C}$. This implies assumption 2.2 a).

If we endow X with the norm

$$\|(x, v)\| := \max\{\|x\|, \sup_{-\tau \leq r \leq 0} \|v(r)\|\},$$

we find that

$$\|(x, v)\| \leq \frac{1}{\lambda}\|(\tilde{x}, \tilde{v})\|.$$

This implies the Hille&Yosida estimates.

Though the theory of functional differential equations is well developed (see, e.g., [H1]) we expect that embedding functional differential equations into our approach will add new aspects to the theory. For an alternative approach using dual semigroup theory see [D4].

6. Abstract functional differential equations whith non-densely defined generators

We extend the idea of the last section in order to show that the class of Lipschitz perturbations of non-densely defined generators is invariant under the introduction of delays, i.e. an abstract functional differential equation with a non-densely defined generator can be reformulated as a Lipschitz perturbation of a non-densely defined generator on a new Banach space.

We consider the abstract semilinear functional differential equation

$$(\diamondsuit\diamondsuit) \qquad \begin{aligned} \frac{d}{dt}u(t) &= Au(t) + F(t)u_t,\ t > t_0 \geq 0; \\ u(t_0 + r) &= v_0(r),\ -\tau \leq r \leq 0 \end{aligned}$$

in a Banach space X. Here $u_t \in C\big([-\tau, 0], X]\big)$ is defined by

$$u_t(r) = u(t + r),\ t \geq t_0, -\tau \leq r \leq 0.$$

Assumptions 6.1. a) A is a closed linear operator on the Banach space X such that $(\lambda - A)$ has a bounded linear inverse on X and

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}$$

for all $n \in \mathbf{N}, \lambda > \omega$ with appropriate real constants M, ω.

b) A is not necessarily densely defined, however. Let $X_0 = \overline{D(A)}$. We assume that the initial values $v_0(r)$ in $(\diamondsuit\diamondsuit)$ are elements in X_0, and we look for solutions u of $(\diamondsuit\diamondsuit)$ with values in X_0. Actually we are interested in finding a solution u with values in

$$C_0 = \tilde{C} \cap X_0$$

with $\tilde{C}$ being a closed convex subset of the Banach space X. Note that C_0 is a closed convex subset, too. So we assume that

$$(6.1) \qquad v_0 \in C\big([\tau, 0], C_0\big) =: C_\tau.$$

c) We assume the following properties of the operators

$$F(t) : C_\tau \to X.$$

(i) For any $v \in C_\tau$, $F(t)v$ is a continuous function of $t \geq 0$.

(ii) For any $t \geq 0, v \in C_\tau$ there exist $\delta, \Lambda > 0$ such that

$$\|F(s)w - F(s)z\| \leq \Lambda \|w - z\|, \tag{6.2}$$

if $t \leq s \leq t + \delta,\ w, z \in C_\tau,\ \|w - v\|, \|z - v\| \leq \delta$.

(iii) For any $\sigma > 0$ there exists some $c > 0$ such that

$$\|F(t)v\| \leq c(1 + \|v\|)$$

if $0 \leq t \leq \sigma, v \in C_\tau$.

Remarks: This problem has the special feature that A is not densely defined in X, but that $F(t)$ may be only defined on a subset of $C([\tau, 0], X_0)$ with $X_0 = \overline{D(A)}$ and map into X instead of X_0.

The assumptions above are not yet sufficient to guarantee that the solution is going to stay in C_0. To this end we assume that $\tilde{C}$ is invariant under $\lambda(\lambda - A)^{-1}$ and that F satisfies a *subtangential condition.*

Assumptions 6.2. a) $\lambda(\lambda - A)^{-1}$ maps $\tilde{C}$ into itself for sufficiently large $\lambda > \omega$.

b) For $t \geq 0, v \in C_\tau$

$$\frac{1}{h} dist\Big(v(0) + hF(t)v; \tilde{C}\Big) \to 0 \quad as \quad h \downarrow 0.$$

Here

$$dist(z; \tilde{C}) = \inf_{x \in \tilde{C}} \|z - x\|$$

gives the distance of a point $z \in X$ from the set $\tilde{C}$.

In general we cannot solve ($\diamondsuit\diamondsuit$) in this strong formulation, if $v_0(r) \in C_0 \setminus D(A)$. So, for arbitrary $v_0 \in C_\tau$, we solve it in the integrated form

$$(\heartsuit\heartsuit) \qquad \begin{aligned} u(t) &= v_0(0) + A\int_{t_0}^{t} u(s)ds + \int_{t_0}^{t} F(s)u_s ds,\ t \geq t_0. \\ u(t_0 + r) &= v_0(r),\ -\tau \leq r \leq 0. \end{aligned}$$

A solution to (♡♡) is called an *integral solution* to (◇◇).

Theorem 6.3. *Let the assumptions 6.1 and 6.2 be satisfied. Then there exists a unique continuous solution to* (♡♡) *with values in* C_0.

A more general version of theorem 6.3 is presented next:

Theorem 6.4. *Let the assumptions 6.1 be satisfied. Then there exists a continuous solution to* (♡♡) *for any initial condition* $v_0 \in C_\tau$ *at time* $t_0 \geq 0$ *iff*

$$\frac{1}{h} dist\Big(T_0(h)v(0) + S(h)F(t)v; C_0\Big) \to 0$$

for $h \downarrow 0, t \geq t_0, v \in C_\tau$. *The solution (if it exists) is unique.*

In theorem 6.4, T_0 denotes the strongly continuous semigroup on X_0 generated by the part A_0 of A in X_0 and S the 'integrated semigroup' generated by A. See section 1.

In order to prove these two theorems we could use the variation of constants formula (1.5). See theorem 1.3 and corollary 1.6. (♡♡) can be equivalently written in the form

(♠♠)
$$u(t) = T_0(t - t_0)v_0(0) + \lim_{\lambda\to\infty} \int_{t_0}^t T_0(t - s)\lambda(\lambda - A)^{-1}F(s)u_s ds, \ t \geq t_0;$$
$$u(t_0 + r) = v_0(r), \ -\tau \leq r \leq 0.$$

If we choose this approach (compare [M2]) we will still have the hassle of showing that the solutions induce a semiflow on C_τ. Alternatively we reformulate the problem as a problem without delay but on a larger Banach space and apply theorem 2.3 and 2.11.

We note that, at least formally,

$$(\partial_t - \partial_r)u_t(r) = 0; \ -\tau < r < 0;$$

$$u_t(0) = u(t).$$

Setting $w(t) = (u(t), u_t)$ we can write (◇◇) in the form

(◇)
$$\frac{d}{dt}w(t) = \mathcal{A}w(t) + G(t)w(t), \ t > t_0 \geq 0;$$
$$w(t_0) = (v_0(0), v_0)$$

with

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{dr} \end{pmatrix}$$

and

$$G(t)\big(v(0), v\big) = \big(F(t)v, 0\big), \quad v \in C_\tau.$$

We choose

$$\mathcal{X} = X \times C\big([\tau, 0], X_0\big),\ \mathcal{X}_0 = \{(v(0), v), v \in C\big([\tau, 0], X_0\big)\},$$

$$\mathcal{C} = \tilde{C} \times C_\tau.$$

Note that C_τ can be identified with $\mathcal{C}_0 := \mathcal{C} \cap \mathcal{X}_0$ and $\mathcal{X}_0$ with $C\big([\tau, 0], X_0\big)$. Further

$$D(\mathcal{A}) = \{(v(0), v); v \in C^1([-\tau, 0], X_0),\ v(0) \in D(A)\}.$$

Note that $\overline{D(\mathcal{A})} = \mathcal{X}_0$.

Let us calculate the resolvent of $\mathcal{A}$. To this end we solve the equation

$$(\lambda - \mathcal{A})(x, v) = (\tilde{x}, \tilde{v}),$$

i.e.

$$(\lambda - A)x = \tilde{x}, \quad \lambda v(r) - \frac{d}{dr}v(r) = \tilde{v}(r),\ -\tau < r < 0; \quad v(0) = x.$$

The solution is given by

$$x = (\lambda - A)^{-1}\tilde{x}, \quad v(r) = xe^{\lambda r} + \int_r^0 e^{\lambda(r-s)}\tilde{v}(s)ds,\ -\tau \leq r \leq 0.$$

Obviously $(\lambda - \mathcal{A})^{-1}$ is a bounded everywhere defined operator for large enough $\lambda > 0$. In order to estimate its norm we renormalize the space X equivalenty such that

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda - \omega} \tag{6.3}$$

for $\lambda > \omega$. See [P2], lemma 1.5.1. This renormalization has also the consequence that

$$\|T_0(t)\| \leq e^{\omega t} \tag{6.4}$$

for $t \geq 0$. We endow $\mathcal{X}$ with the norm

$$\|(x, v)\| = \max(\|x\|, \|v\|).$$

We find

$$\|x\| \leq \frac{1}{\lambda - \omega}\|\tilde{x}\|$$

and

$$\|v\| \leq \|x\| e^{\lambda r} + \frac{1}{\lambda}(1 - e^{\lambda r})\|\tilde{v}\| \leq \frac{1}{\lambda - \omega}\|(\tilde{x}, \tilde{v})\|,$$

if we assume $\omega > 0$ without restricting the generality. Hence

$$\|(x, v)\| \leq \frac{1}{\lambda - \omega}\|(\tilde{x}, \tilde{v})\|.$$

Let us check whether $\lambda(\lambda - \mathcal{A})^{-1}$ leaves $\mathcal{C}$ invariant, if $\lambda(\lambda - A)^{-1}$ leaves C invariant. Obviously $\lambda x \in C_0$ and also

$$\lambda v(r) = \lambda x e^{\lambda r} + \lambda \int_r^0 e^{\lambda(r-s)} \tilde{v}(s) ds \in C_0$$

because

$$v(s) \in C_0, \quad \lambda e^{\lambda r} + \lambda \int_r^0 e^{\lambda(r-s)} ds = 1.$$

The other assumptions of theorem 2.3 and 2.11 are easily checked. Note that, for checking the subtangential conditions only the first coordinates of the semigroup and the 'integrated semigroup' are needed because of the special form of G. They are given by T_0 and S because the first coordinate of $(\lambda - \mathcal{A})^{-1}(\tilde{x}, \tilde{v})$ is $(\lambda - A)^{-1}\tilde{x}$. Recall the relations between resolvent and the Laplace transforms of the semigroup and 'integrated semigroup' in remark 1.10.

Theorem 2.3 or 2.11 respectively provide unique continuous integral solutions to ($\diamondsuit$), i.e. to

$$\begin{aligned} u(t) &= v_0(0) + A \int_{t_0}^t u(s) ds + \int_{t_0}^t F(s) v(s) ds, \ t \geq t_0. \\ u(t_0 + r) &= v_0(r), \ -\tau \leq r \leq 0. \\ v(t, r) &= v_0(r) + \frac{d}{dr} \int_{t_0}^t v(s, r) ds; \ t \geq t_0, -\tau \leq r \leq 0. \end{aligned}$$

The last two equations are uniquely solved by $v(t) = u_t$. Hence we have found a unique continuous solution to ($\heartsuit\heartsuit$). The variation of constants formula (1.5) implies that it also solves ($\spadesuit\spadesuit$) uniquely. Knowing this relation may be important to prove regularity results, in particular if T_0 is an analytic semigroup.

We leave it to the reader to translate the results in section 3 and 4 to ($\diamondsuit\diamondsuit$). Note that this approach works in particular if A generates a strongly continuous semigroup. So one can rediscover the stability results in [P1], e.g..

7. Age-dependent population dynamics

We consider the following general version of model (1) in the introduction of part I:

$$
\begin{aligned}
(\partial_t + \partial_a) n(t,a) &= F_1(n(t.,\cdot))(a),\ 0 < a < \infty, \\
n(t,0) &= F_0(n(t,\cdot)),\ t > 0, \\
n(0,a) &= n_0(a), a \geq 0
\end{aligned}
\tag{7.1}
$$

with a given function n_0 on $[0,\infty)$. In order to keep the presentation somewhat simpler we have restricted ourselves to the time-autonomous case.

As we want to include systems of populations we allow $n(t,a)$ to take values in $\mathbf{R}^m$, m a natural number. The natural space for the age distribution is

$$n(t,\cdot) \in L_1([0,\infty), \mathbf{R}^m) =: L_1$$

because the vector $\int_0^\infty n(t,a)da$ gives the total population sizes of the populations involved in the system. For some biological reason (see the model in the introduction) we would like our solutions to stay in a closed convex set

$$C_0 \subseteq L_1 = L_1([0,\infty), \mathbf{R}^m).$$

Hence we assume

$$n_0(a) \in C_0 \text{ for a.a. } a > 0. \tag{7.2}$$

Further we make the following

Assumptions 7.1.

a) F_1 maps the closed convex subset C_0 of L_1 into L_1 and satisfies a Lipschitz condition

$$\|F_1(x) - F_1(y)\|_1 \leq \Lambda \|x - y\|_1$$

for $x, y \in C_0$.

b) F_0 maps C_0 into $\mathbf{R}^m$ and satisfies a Lipschitz condition

$$\|F_0(x) - F_0(y)\| \le \Lambda \|x - y\|_1$$

for $x, y \in C_0$.

Here $\|\cdot\|_1$ denotes the L_1-norm

$$\|x\|_1 = \int_0^\infty \|x(a)\| da.$$

The global Lipschitz condition can be replaced by a local one if only the results in section 2 shall be applied.

The classical approach to such a problem consists in integrating the partial differential equation in (7.1) along characteristic curves and reducing the problem to an integral equation. See [W1]. We show how this problem can be embedded into our theory. In particular we will find that (7.1) can be solved in a very natural generalized sense for initial data which are measurable and satisfy (7.2) for a.a. $a \ge 0$.

We choose $X = \mathbf{R}^m \times L_1$ and $X_0 = \{0\} \times L_1$.Note that X_0 can be identified with L_1. Further we set $C = \mathbf{R}^m \times C_0$. Thus C_0 can be identified with $C \cap X_0$.

The operator A is defined on $D(A) = AC[0, \infty)$, the space of absolutely continuous functions with values in $\mathbf{R}^m$,

$$Ax = (-x(0), -x'), x \in D(A). \tag{7.3}$$

Note that $\overline{D(A)} = X_0$. The nonlinearity $F : C_0 \to X$ is defined by

$$Fx = (F_0 x, F_1 x) \tag{7.4}$$

Note, that if $x \in D(A)$, we have $Ax + Fx \in X_0$ if and only if $x(0) = F_0(x)$.

If $r \in \mathbf{R}^m, y \in L_1$, then, for $\lambda > 0$,

$$(\lambda - A)^{-1}(r, y) = (0, x) \tag{7.5}$$

with

$$x(a) = re^{-\lambda a} + \int_0^a e^{\lambda(\tau - a)} y(\tau) d\tau. \tag{7.6}$$

If we endow X with the norm

$$\|(r, x)\| = \|r\| + \int_0^1 \|x(a)\| da$$

we find that

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda}.$$

The other assumptions in 2.1 are easily checked. Note that it is difficult to find an assumption which makes asssumption 2.2 a) apply. So we need to apply theorem 2.11 instead of theorem 2.3. To this end we determine the semigroup T_0 generated by A_0, the part of A in X_0 and the 'integrated semigroup' S generated by A. Now

$$D(A_0) = \{x \in AC[0,\infty), x(0) = 0\},\ A_0 x = -x'.$$

Hence

$$T_0(t)x(a) = \begin{cases} x(a-t) & \text{if } a > t \\ 0 & \text{if } a \leq t \end{cases} \tag{7.7}$$

This can be most easily seen from (7.5), (7.6) with $r = 0$ and the fact that

$$(\lambda - A_0)^{-1} = \int_0^\infty e^{-\lambda t} T_0(t) dt.$$

From (7.5), (7.6), and (1.10) we find that

$$S(t)(r, x) = (0, y(t)) \tag{7.8}$$

with

$$y(t)(a) = rH(t-a) + \int_0^t x(a-s)H(a-s)ds \tag{7.9}$$

and H denoting the Heaviside function $H(a) = 1, a \geq 0, H(a) = 0, a < 0$. Hence

$$T_0(h)x + S(h)Fx = (0, z(h))$$

with

$$\begin{aligned} z(h)(a) &= x(a-h)H(a-h) + \int_0^h F_1 x(a-s)H(a-s)ds + F_0 x H(h-a) \\ &= x(a-h)H(a-h) + hF_1 x(a-h)H(a-h) + F_0 x H(h-a) \\ &+ h(F_1 x(a) - F_1 x(a-h)H(a-h)) + \int_0^h (F_1 x(a-s)H(a-s) - F_1 x(a))ds. \end{aligned}$$

Define

$$Z(h)(x)(a) = x(a-h)H(a-h) + hF_1 x(a-h)H(a-h) + F_0 x H(h-a) \tag{7.10}$$

Then

$$\frac{1}{h}\|z(h) - Z(x)(h)\|_1 \leq \int_0^\infty |F_1x(a) - F_1x(a-h)H(a-h)|da$$

$$+ \int_0^\infty \int_0^1 |F_1x(a-sh)H(a-sh) - F_1x(a)|dads \ \to 0, \quad h \downarrow 0.$$

Thus, by lemma 2.4 b), we have that

$$\frac{1}{h}dist(T_0(h)x + S(h)x; C_0) \to 0, \quad h \downarrow 0$$

if and only if

$$\frac{1}{h}dist(Z(h)x; C_0) \to 0, \quad h \downarrow 0$$

Proposition 2.11 now provides us with a solution of (1.3), i.e. we have the following result:

Theorem 7.2. *Let the assumptions 7.1 be satisfied and*

$$\frac{1}{h}dist(Z(h)x; C_0) \to 0, \quad h \downarrow 0, \tag{7.11}$$

for any $x \in C_0$. Let $n_0 \in C_0$. Then there exists a uniquely determined continuous function $n(t)$ with values in C_0, which satisfies (7.1) in the following sense: For any $t > 0$, $\int_0^t n(s,a)ds$ is absolutely continuous in a and

$$n(t,a) - n_0(a) + \partial_a \int_0^t n(s,a)ds = \int_0^t F_1(n(s,\cdot))(a)ds,$$

$$\int_0^t n(s,0)ds = \int_0^t F_0(n(s,\cdot))ds.$$

(7.11) is also necessary for any solution starting in C_0 to stay in C_0.

Recall that Z has been defined in (7.10).

Our example in the introduction becomes a special case of (7.1) by setting

$$F_0x = \beta \int_0^1 x(a)(1-x(a))da;$$
$$F_1x(a) = \alpha x(a)(1-x(a)), \qquad 0 \leq a \leq 1;$$
$$F_1x(a) = 0, \quad a > 1$$

with $C_0 = \{x \in L_1([0,1], \mathbf{R});\ 0 \le x \le 1\ a.e.\}$. In order to check (7.11) note that

$$x(a-h)H(a-h) + hF_1 x(a-h)H(a-h) = H(a-h)x(a-h)(1 + h\alpha(1 - x(a-h)))$$

is monotone increasing in $x(a-h) \in [0,1]$, hence this term lies in $[0,1]$ if $x \in C_0$ and h is sufficiently small. Note further that

$$0 \le F_0 x \le 1 \text{ if} \beta \le 4, x \in C_0.$$

Hence, we have $Z(h)x \in C_0$, if $\beta \le 4, x \in C_0, h$ sufficiently small.

Theorem 7.2 now provides us with the following result:

Corollary 7.3. Let $n_0 \in L_1[0,1]$, $0 \le n_0 \le 1$ a.e.. Let $\beta \le 4$. Then there exists a uniquely determined continuous function $n(t)$ with values in $L_1[0,1]$, $0 \le n(t,a) \le 1$ for a.a. $a \in [0,1]$, which satisfies (7.1) in the following sense: For any $t > 0$, $\int_0^t n(s,a)ds$ is absolutely continuous in a and

$$n(t,a) - n_0(a) + \partial_a \int_0^t n(s,a)ds = \alpha \int_0^t n(s,a)(1 - n(s,a))ds,$$

$$\int_0^t n(s,0)ds = \beta \int_0^t \int_0^1 n(s,a)(1 - n(s,a))ds.$$

8. Age-structured functional differential equations

We consider the following general version of model (2) in the introduction of part I:

$$\begin{aligned}(\partial_t + \partial_a)n(t,a) &= F_1(n_t)(a),\ 0 < a < \infty,\\ n(t,0) &= F_0(n_t),\ t > 0,\\ n(t,a) &= n_0(t,a), -\tau \le t \le 0,\ a \ge 0\end{aligned} \tag{8.1}$$

with a given function u_0 on $[-\tau, 0] \times [0, \infty)$. Here $n_t(r,a) = n(t+r, a)$ for $-\tau \le r \le 0, a \ge 0$. It is only for simplicity that we have restricted (8.1) to the time-autonomous case.

Again we want to include systems of populations, so we allow $n(t,a)$ to take values in $\mathbf{R}^m$, m a natural number and, as in section 7, the natural space for the age distribution is

$$n(t,\cdot) \in L_1([0,\infty), \mathbf{R}^m) =: L_1$$

because the vector $\int_0^\infty n(t,a)da$ gives the total population sizes of the populations involved in the system. For some biological reason (see the model in the introduction) we would like our solutions to stay in a closed convex set

$$C_0 \subseteq L_1 = L_1([0,\infty), \mathbf{R}^m).$$

Hence we set

$$C_\tau = C([\tau, 0], C_0)$$

and make the following

Assumptions 8.1.

a) F_1 maps the closed convex subset C_τ into $C([-\tau,0], L_1)$ and satisfies a Lipschitz condition

$$\|F_1(x) - F_1(y)\|_0 \leq \Lambda \|x - y\|_0$$

for $x, y \in C_\tau$.

b) F_0 maps C_τ into $\mathbf{R}^m$ and satisfies a Lipschitz condition

$$\|F_0(x) - F_0(y)\| \leq \Lambda \|x - y\|_0$$

for $x, y \in C_\tau$.

Here $\|\cdot\|_0$ denotes the norm

$$\|x\|_0 = \sup_{-\tau \leq t \leq 0} \int_0^\infty \|x(t,a)\| da.$$

The global Lipschitz condition can be replaced by a local one if only the results in section 2 shall be applied.

A possible approach to this problem consists in integrating the partial differential equation in (8.1) along characteristic curves and reducing the problem to an integral equation. But proving that

$$t \mapsto n_t$$

defines a dynamical system on C_τ becomes awkward. Alternatively we embed this problem into the framework of section 6.

As in section 7 we choose $X = \mathbf{R}^m \times L_1$ and $X_0 = \{0\} \times L_1$. Note that X_0 can be identified with L_1. Further we set $C = \mathbf{R}^m \times C_0$. Thus C_0 can be identified with $C \cap X_0$. We define A as in (7.3) and the nonlinearity $F : C_\tau \to X$ by

$$Fx = (F_0 x, F_1 x) \tag{8.2}$$

Combining the consideration in section 7 with theorem 6.4 we define

(8.3) $Z(h)(x)(a) = x(0, a-h)H(a-h) + h(F_1 x)(a-h)H(a-h) + F_0 x H(h-a), \ x \in C_\tau.$

Theorem 6.4 now yields the following result:

Theorem 8.2. *Let the assumptions 8.1 be satisfied and*

$$\frac{1}{h} dist(Z(h)x; C_0) \to 0, \quad h \downarrow 0, \tag{8.4}$$

for any $x \in C_\tau$. Let $t \mapsto n_0(t, \cdot) \in C_\tau$. Then there exists a uniquely determined continuous function $n(t)$ with values in C_0, which satisfies (8.1) in the following sense: For any $t > 0$, $\int_0^t n(s,a)ds$ is absolutely continuous in a and

$$n(t,a) - n_0(a) + \partial_a \int_0^t n(s,a)ds = \int_0^t F_1(n_s)(a)ds,$$

$$\int_0^t n(s,0)ds = \int_0^t F_0(n_s)ds.$$

The mapping $t \mapsto n_t$ defines a dynamical system on C_τ. (8.4) is also necessary for any solution starting in C_τ to stay in C_τ.

9. Abstract semilinear boundary value evolution problems

Following Greiner [G2, G3] we consider a semilinear evolution problem of the following form:

$$\frac{d}{dt}u(t) = Au(t) + F_1(t)u(t), \quad Lu(t) = F_0(t)u(t), \quad t > t_0; \quad u(t_0) = y_0. \tag{9.1}$$

$Lu(t) = F_0(t)u(t)$ constitutes the semilinear boundary condition.

Assumptions 9.1.

a) A is a densely defined linear operator on a Banach space X_0.

b) L is a linear surjection from D(A) to a Banach space Z.

c) The restriction A_0 of A to the kernel *Ker* L of L is the generator of a strongly continuous semigroup.

d) $\|Ly\| \geq \gamma(\lambda - \tilde{\omega})\|y\|$ for $y \in \text{Ker }(\lambda - A),\ \lambda > \tilde{\omega}$ with some constants $\gamma, \tilde{\omega} \in \mathbf{R}$.

Actually we are interested in finding a solution u with values in a closed convex subset C_0 of X_0.

e) We assume the following properties of the operators

$$F_0(t) : C_0 \to Z, F_1(t) : C_0 \to X_0.$$

(i) For any $y \in C_0$, $F_j(t)y$ is a continuous function of $t \geq 0$.

(ii) For any $t \geq 0, x \in C_0$ there exist $\delta, \Lambda > 0$ such that

$$\|F_j(s)y - F_j(s)z\| \leq \Lambda\|y - z\|,$$

if $t \leq s \leq t + \delta,\ y, z \in C_0,\ \|y - x\|, \|z - x\| \leq \delta$.

(iii) For any $\tau > 0$ there exists some $c > 0$ such that

$$\|F_j(t)x\| \leq c(1 + \|x\|)$$

if $0 \leq t \leq \tau, x \in C_0$.

Assumption 9.1 d) has been adopted from Greiner [G2], theorem 2.1. Actually the assumptions 9.1 are not sufficient for C_0 to be invariant under the solutions to (9.1). We still have to require subtangential conditions in the spirit of assumption 2.2 or, more generally, of theorem 2.11. Before we do so we show how problem (9.1) can be embedded into the general theory presented in section 2.

Following Kellermann [K1] we choose

$$X = Z \times X_0$$

and define an operator $\mathcal{A}$ in X by

$$D(\mathcal{A}) = \{0\} \times D(A), \quad \mathcal{A}(0, y) = (-Ly, Ay). \tag{9.2}$$

Note that $\overline{D(\mathcal{A})} = \{0\} \times X_0$ can (and will) be identified with X_0. Further we identify the closed convex set C_0 in X_0 with the closed convex set $\{0\} \times C_0$.

Finally we define $F : \{0\} \times C_0 \to X$ by

$$F(t)(0, y) = (F_0(t)y, F_1(t)y),\ y \in C_0.$$

Notice that a solution u to $\frac{d}{dt}u(t) = \mathcal{A}u(t) + F(t)u(t)$ with values in C_0 automatically satisfies (9.1) and vice versa.

The assumptions 2.1 concerning F follow immediately. The assumptions 2.1 concerning $\mathcal{A}$ are a consequence of the following

Lemma 9.2. a) $X_0 = \mathrm{Ker}L \oplus \mathrm{Ker}(\lambda - A)$ *for any* λ *in the resolvent set of* A_0.
b) *The part* $\mathcal{A}_0$ *of* $\mathcal{A}$ *in* X_0 *is* A_0 *where we have identified* X_0 *with* $\{0\} \times X_0 = \overline{D(\mathcal{A})}$. *In particular* $\mathcal{A}_0$ *generates a strongly continuous semigroup on* X_0 *which can be identified with the semigroup generated by* A_0.
c) *There exist* $\omega \in \mathbf{R}, M > 0$ *such that* $\lambda - \mathcal{A}$ *is invertible and*

$$\|(\lambda - \mathcal{A})^{-1}\| \leq \frac{M}{\lambda - \omega}$$

for $\lambda > \omega$. *Moreover*

$$(\lambda - \mathcal{A})^{-1}(z, y) = (0, (\lambda - A_0)^{-1}y + L_\lambda z)$$

for $z \in Z, y \in X_0$ *with* L_λ *denoting the inverse of the restriction of* L *to* $\mathrm{Ker}(\lambda - A)$. *See a).*

Indeed, as $(\lambda - \mathcal{A})^{-1}$ maps X into X_0 the estimates in assumption 1.1 now follow immediately from the Hille&Yosida theorem and the fact that $(\lambda - \mathcal{A})^{-(n+1)} = (\lambda - \mathcal{A}_0)^{-n}(\lambda - \mathcal{A})^{-1}$.

Proof of lemma 9.2. a) See the proof of lemma 1.2 in Greiner [G1].
b) If $(0, y) \in D(\mathcal{A}_0)$, then, by (9.2), $y \in D(A)$ and $Ly = 0$, thus $y \in D(A_0)$ and $\mathcal{A}_0(0, y) = (0, A_0 y)$. The rest is obvious.
c) First we realize that $\lambda - \mathcal{A}$ is injective if λ is in the resolvent set of A_0. Indeed, if $(\lambda - \mathcal{A})(0, y) = (0, 0)$, then, by (9.2), $y \in D(A_0)$ and $(\lambda - A_0)y = 0$. Hence $y = 0$.
Next we look for $y \in D(A)$ such that

$$(\lambda - \mathcal{A})(0, y) = (z, \tilde{y})$$

for given $(z, \tilde{y}) \in X$. As $\lambda - A_0$ is surjective for λ being in the resolvent set of A_0 we find $y_0 \in \mathrm{Ker}L$ such that

$$(\lambda - A_0)y_0 = \tilde{y}.$$

Further, as L is surjective by assumption 9.1 b), we find $\hat{y} \in D(A)$ such that

$$L\hat{y} = z.$$

Actually, by a), we can arrange that $\hat{y} \in \mathrm{Ker}(\lambda - A)$. Hence

$$(\lambda - A)(y_0 + \hat{y}) = \tilde{y}, \quad L(y_0 + \hat{y}) = z, \quad y_0 \in \mathrm{Ker}L,\ \hat{y} \in \mathrm{Ker}(\lambda - A).$$

By assumption 9.1 d) we have

$$\|z\| \geq \gamma(\lambda - \tilde{\omega})\|\hat{y}\|$$

and, from the Hille&Yosida theorem,

$$\|y_0\| \leq \frac{\hat{M}}{\lambda - \omega}\|\tilde{y}\|.$$

Hence

$$\|y_0 + \hat{y}\| \leq \frac{\hat{M}}{\lambda - \hat{\omega}}\|\tilde{y}\| + \frac{1/\gamma}{\lambda - \tilde{\omega}}\|z\| \leq \frac{M}{\lambda - \omega}(\|\tilde{y}\| + \|z\|).$$

for $\lambda > \omega$ with $M = \max(\hat{M}, 1/\gamma)$, $\omega = \max(\hat{\omega}, \tilde{\omega})$.

Kellermann [K1] has already observed that $\mathcal{A}$ generates an 'integrated semigroup' under a weaker condition than assumption 9.1 d). Assumption 9.1 d) provides the additional property that the 'integrated semigroup' is locally Lipschitz in the operator norm topology. The assumptions 9.1 imposes stronger assumptions on A and L but weaker assumptions on F_0 than the assumptions in Greiner [G3]. In particular we can dispense with hypothesis **H3** in [G3]. Greiner [G2], after theorem 2.1, gives examples in which assumption 9.1 d) is satisfied. In particular the age-dependent population model in section 7 is a special case with

$$D(A) = AC[0, \infty), Ay = -y', \quad Ly = y(0),\ y \in D(A),\ Z = \mathbf{R}^m.$$

For, if $y \in \mathrm{Ker}(\lambda - A)$, then $y(a) = y(0)e^{-\lambda a}$, hence $\|y\| = \int_0^\infty \|y(a)\|da \leq \frac{1}{\lambda}\|y(0)\|$, i.e. condition 9.1 d) holds.

In the special case in section 7 we were able to give a concrete subtangential condition which guaranteed the forward invariance of a closed convex set. This is more difficult for the abstract boundary value problem. Recall that the subtangential condition derived in lemma 2.9 is sufficient and necessary. Using lemma 9.2 this condition translates into

$$\text{(9.3)} \qquad \frac{1}{h} dist\left(T_0(h)y + \int_0^h T_0(s)F_1(t)yds + \lim_{\lambda\to\infty}\int_0^h T_0(s)\lambda L_\lambda F_0(t)yds;\ C_0\right) \to 0$$

for $h \downarrow 0, t \geq t_0$. Here T_0 is the strongly continuous semigroup generated by A_0. A similar proof as for lemma 2.9 yields the following sufficient subtangential condition:

$$(9.4) \qquad \frac{1}{h} \liminf_{\lambda \to \infty} dist(y + hF_1(t)y + \lambda L_\lambda F_0(t)y; C_0) \to 0, \quad h \downarrow 0, t \geq t_0, y \in C_0.$$

Solutions to equation (♡) in section 2 with $\mathcal{A}$ replacing A translate into integral (or mild) solutions solutions to (9.1):

$$(9.5) \;\; u(t) = y_0 + A \int_{t_0}^t u(s)ds + \int_{t_0}^t F_1(s)u(s)ds, \quad L \int_{t_0}^t u(s)ds = \int_{t_0}^t F_0(s)u(s)ds, \; t \geq t_0$$

with the implicit understanding that $\int_{t_0}^t u(s)ds \in D(A)$. Compare Greiner [G3], definition 1.1.

Theorem 9.3. *Let the assumptions 9.1 be satisfied. Assume that the subtangential conditions (9.3) or (9.4) holds. Then there exists a unique solution u to (9.5) with values in C_0.*

We mention that the solutions to (9.5) satisfy the variation of constants formula

$$u(t) = T_0(t - t_0) + \int_{t_0}^t T_0(t-s)F_1(s)u(s)ds + \lim_{\lambda \to \infty} \int_{t_0}^t T_0(t-s)\lambda L_\lambda F_0(s)u(s)ds, \quad t \geq t_0.$$

Here T_0 is the strongly continuous semigroup generated by the operator A_0 in assumption 9.1 c). This formula follows from (♠) in section 2 and lemma 9.2 c).

The results in section 3 and section 4 can readily be translated. In particular one obtains similar (in)stability results for steady states as Greiner [G3].

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