Semiflows Generated by Lipschitz Perturbations of Non-densely Defined Operators

I. The Theory

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Abstract

A variety of problems in differential equations ((abstract) functional differential equations, age-dependent population models (with and without delay), evolution equations with boundary conditions e.g.) can be written as semilinear Cauchy problems with a Lipschitz perturbation of a closed linear operator which is not non-densely defined but satisfies the estimates of the *Hille&Yosida* theorem. A natural generalized notion of solution is provided by the *integral solutions* in the sense of *Da Prato&Sinestrari*. Ideas from 'integrated semigroup' theory yield a variation of constants formula which allows to transform the integral solutions of the evolution equation to solutions of an abstract semilinear Volterra integral equation. The latter can be used to find integral solutions to the Cauchy problem; moreover one finds sufficient and necessary conditions for the (forward) invariance of closed convex sets under the solution flow. The solution flow can be shown to form a dynamical system. Conditions for the regularity of the flow in time and initial state are derived. The steady states of the flow are characterized and sufficient conditions for local stability and instability are found. Finally the problems mentioned at the beginning are fitted into the general framework.

Key Words: Hille&Yosida theorem, Cauchy problems, evolution equations, integral solutions, semigroups,'integrated semigroups', variation of constants formula, abstract Volterra integral equation, dynamical systems, invariance of closed convex sets, regularity in initial value and time, (linear and nonlinear) generator, steady states, local (asymptotic) stability, semilinear boundary conditions, (abstract) functional differential equations, age-structured population models (with and without delays)

Introduction

Semilinear Cauchy problems

$$\frac{d}{dt}u(t) = Au(t) + Fu(t),\ u(0) = x_0 \tag{1}$$

with F being a Lipschitz perturbation of the generator A of a strongly continuous semigroup are well understood. See, e.g., [P2], section 6.1. Quite often one faces nonlinear Cauchy problems, however, which cannot be split up in this way because a nonlinearity appears in the domain of A though the action of A itself is linear. In this case it is sometimes possible to rewrite the problem as a semilinear problem (1) by removing the nonlinearity from the domain of A and incorporating it into the Lipschitz perturbation F. The prize to be paid (according to a universal principle of conservation of difficulty) consists in ending up in a larger Banach space (which is not so bad) and with a linear operator which is not densely defined.

In order to illustrate this phenomenon we consider the following problem in age-structured population dynamics:

$$\begin{aligned}
(\partial_t + \partial_a)n(t,a) &= \alpha n(t,a)\Big(1 - n(t,a)\Big), \quad t > 0, 0 < a < 1, \\
n(t,0) &= \beta \int_0^1 n(t,a)\Big(1 - n(t,a)\Big)da, \quad t > 0, \\
n(0,a) &= n_0(a), \quad a \geq 0
\end{aligned} \tag{2}$$

with a given function n_0 on $[0,1]$. α, β are positive constants. This model can be considered a simplistic description of the growth of a plant population which reproduces both by layers (the terms in the right hand side of the first equation) and by seeds (second equation). Here we have assumed intra age-group competition, i.e. the plants only hamper other plants of the same age. Since the seed production term in the second equation should be non-negative in oder to make biological sense, we look for solutions n to (2) with

$$0 \leq n(t,a) \leq 1. \tag{3}$$

The classical approach to such a problem consists in integrating the partial differential equation in (2) along characteristic curves and reducing the problem to an integral equation. See [W1]. Alternatively we try to deal with this problem as a Cauchy problem, i.e. as an abstract differential equation. The natural state space of an age-structured population model is the space of integrable functions, because the L_1-norm can be interpreted as population size. Hence we set $u(t) = n(t, .) \in L_1[0, 1] =: L_1$. Then we rewrite (2) as

$$\frac{d}{dt}u = Gu$$

with

$$Gu = -u' + F_1 u, \; D(G) = \{u \in AC[0, 1], u(0) = F_0 u\}, \tag{3}$$

and

$$\begin{aligned} F_0 x &= \beta \int_0^1 n(t, a)\Big(1 - n(t, a)\Big)da, \\ (F_1 x)(a) &= \alpha n(t, a)\Big(1 - n(t, a)\Big). \end{aligned} \tag{5}$$

Note that the nonlinearity F_0 has entered the domain of G, whereas the action of G itself is linear in the highest order term. Actually it is well possible to deal with this formulation of the problem by employing the *Crandall&Liggett* theorem (see [C7], [C6].) . The advantage of this approach consists in providing a dyamical system right away. But it is difficult to relate the notion of solution one obtains this way to the Cauchy problem itself unless one preassumes enough regularity or the underlying Banach space is reflexive (or, more generally, satisfies the Radon-Nikodým property). See [C6], section 2.4. Further it is hard to prove results like linearized stability in this framework. The reason for this lies in the fact that the *Crandall-Liggett* theorem does not exploit the semilinear character of the boundary condition because it is taylored for a more general situation. Semilinear boundary conditions can be handled in several ways in an abstract framework. See the work by Amann [A1, A2, A3], Goldstein [G1], Greiner [G2, G3], and Desch, Schappacher and Kang Pei Zang [D3].

Our approach will remove the nonlinearity from the domain of G and incorporate it into the Lipschitz perturbation. To this end we enlarge the state space and choose $X = \mathbf{R} \times L_1$. We are still looking for solutions with values in the space L_1 which can be identified with $X_0 = \{0\} \times L_1$. Actually our solutions ought to take values in $C_0 = \{x \in L_1[0.1]; 0 \leq x \leq 1 \; a.e.\}$. C_0 can be identified with $X_0 \cap C, C = \mathbf{R} \times C_0$.

We define a linear operator A in X by $D(A) = AC[0, 1]$, the space of absolutely continuous functions,

$$Ax = (-x(0), -x'), \; x \in D(A). \tag{4}$$

Note that $\overline{D(A)} = X_0$.

We now rewrite the Cauchy problem in the form (1) with $F : C_0 \to X$ being defined by

$$Fx = (F_0 x, F_1 x). \tag{6}$$

Note, that if $x \in D(A)$, we have $Ax + Fx \in X_0$ if and only if $x(0) = F_0(x)$, i.e. $x \in D(G)$. In this way we have removed the nonlinearity F_0 from the domain of G and have incorporated it in the Lipschitz perturbation F provided that we succeed in solving (1) in X_0. Note that both A and F map out of X_0 and that A is not densely defined in X. Fortunately the following is still true:

If $r \in \mathbf{R}, y \in L_1, \lambda > 0$, then

$$(\lambda - A)^{-1}(r, y) = (0, x) \tag{7}$$

with

$$x(a) = re^{-\lambda a} + \int_0^a e^{\lambda(\tau - a)} y(\tau) d\tau. \tag{8}$$

Hence, if we endow X with the norm

$$\|(r, x)\| = |r| + \int_0^1 |x(a)| da,$$

we find that

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda}.$$

Under this condition Da Prato and Sinestrari [D1] have studied linear homogeneous problems

$$\frac{d}{dt} u(t) = Au(t) + f(t), \; u(0) = x_0 \tag{9}$$

with $x_0 \in X_0 = \overline{D(A)}, f : [0, \infty) \to X$ being locally Bochner integrable and have obtained *integral solutions* to (9) in the following sense (for arbitrary initial data in X_0!):

$$u(t) - x_0 = \int_0^t u(s) ds + \int_0^t f(s) ds, \quad t \geq 0.$$

They have also derived a useful estimate of u in terms of x_0 and f. In section 1 we give an alternative proof of their result using ideas from the theory of 'integrated semigroups' developed by Arendt, Kellermann, Hieber, and Neubrander [A3, A4, A5, K2, K3, N1] and the theory of weakly* continuous semigroups on dual Banach spaces developed by Clément, Diekmann, Gyllenberg, Heijmans, and Thieme [C1, C2, C3, C4, C5]. In particular we derive a variation of constants formula for the integral solutions of (9) which is of own interest. Among other things this formula makes it possible to find sufficient and necessary conditions for the invariance of closed convex subsets of X_0 under the solution flow of (1). See section 2. The properties of the dynamical system generated by the solutions are studied in section 3. There we also derive regularity results. In section 4 we study steady states and their local (asymptotic) stability. In section 3 and 4 we follow the lines of [C3] where the special case that A generates a dual semigroup has been considered. In part II [T2] the unifying power of the theory of Lipschitz perturbations of non-densely defined operators is illustrated. We show in section 5 how functional differential equations fit into this framework. More generally we demonstrate in section 6 that the theory is invariant under the introduction of delays, i.e. an abstract functional differential equation with a non-densely defined linear operator can be rewritten as a Lipschitz perturbation of another non-densely defined linear operator (on a larger space) without memory terms. After studying a more general version of (2) in section 7 we use this idea to study (2) with delay terms in section 8:

$$
\begin{aligned}
(\partial_t + \partial_a) n(t,a) &= \alpha n(t-\sigma,a)\Big(1-n(t,a)\Big), \quad t>0, 0<a<1,\\
n(t,0) &= \beta \int_0^1 n(t-\rho,a)\Big(1-n(t-\rho,a)\Big)da, \quad t>0,\\
n(t,a) &= \phi(t)(a), \quad -\tau \le t \le 0,\ a \ge 0
\end{aligned}
\tag{10}
$$

with a given function $\phi : [-\tau,0] \to L_1[0,1]$. $\alpha, \beta, \sigma, \rho$ are positive constants, $\tau = \max(\sigma,\rho)$.

The notion of *integral solution* allows to solve this problem in the following generalized sense (for arbitray initial data $\phi \in C([-\tau,0], L_1[0,1])$:

Corollary 1. *Let $\phi \in C\Big([-\tau,0], L_1[0,1]\Big)$, $0 \le \phi(t) \le 1$ a.e. on $[0,1]$ for $-\tau \le t \le 0$. Let $\beta \le 4$. Then there exists a uniquely determined continuous function $n(t)$ with values in $L_1[0,1]$, $0 \le n(t,a) \le 1$ for a.a. $a \in [0,1]$, which satisfies (10) in the following sense: For any $t > 0$, $\int_0^t n(s,a)ds$ is absolutely continuous in a and*

$$
n(t,a) - n_0(a) + \partial_a \int_0^t n(s,a)ds = \alpha \int_0^t n(s-\sigma,a)(1-n(s,a))ds,
$$

$$
\int_0^t n(s,0)ds = \beta \int_0^t \int_0^1 n(s-\rho,a)(1-n(s-\rho,a))dsda.
$$

Moreover $(\phi, t) \mapsto n_t$ *defines a continuous dynamical system on* $C_\tau = \{\phi : [-\tau, 0] \to L_1[0,1];\ \phi \text{ continuous},\ 0 \leq \phi(t) \leq 1 \text{ a.e. on } [0,1], -\tau \leq t \leq 0\}$. *Here* $n_t(r) = n(t + r, \cdot)$ *for* $t \geq 0, -\tau \leq r \leq 0$.

Strong solutions to (10) can be found from theorem 3.7.

Corollary 2. *Let the assumptions of corollary 1 be satisfied. Let* ϕ *be differentiable as a function with values in* $L_1[0,1]$ *and* $\phi(0)$ *be absolutely continuous on* $[0,1]$,

$$\phi(0,0) = \beta \int_0^1 \phi(-\rho, a)(1 - \phi(-\rho, a))da.$$

Then the solution n *to (10) provided by corollary 1 is differentiable as a mapping from* $[-\tau, \infty)$ *to* $L_1[0,1]$ *and, for any* $t \geq 0$, $n(t,a)$ *is absolutely continuous in* $[0,1]$ *and (10) holds with the differential equation holding a.e. in* $0 \leq a \leq 1$ *for any* $t > 0$.

In section 9 we show how abstract boundary value problems (see Greiner [G2, G3]) fit into the framework.

More examples of linear operators which are not densely defined but satisfy the estimates of the *Hille& Yosida* theorem can be found in [D1].

The examples and the theory presented in this paper accumulate enough evidence that semilinear Cauchy problems with a Lipschitz perturbation of a linear operator which is not densely defined but satisfies the estimates of the *Hille&Yosida* theorem give rise to a class of nonlinear dynamical systems which is large enough to cover important special cases, but specific enough to have useful additional structure. In future work we plan to study conditions under which the flow generated by (1) has compactifying properties in order to push the door to dynamcial systems theory more open and, e.g., apply the results in section 4 more readily. We also plan to study conditions under which the dynamical system becomes monotone and to use the additional structure to obtain more information than can be obtained by the theory of Hirsch [H3] and Matano [M3]. See also [S1]. As a further application we intend to study abstract age-structured evolution equations, e.g. age-dependent diffusion problems.

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1. Integral solutions to nonhomogeneous Cauchy problems (Da Prato's and Sinestrari's result revisited. 'Integrated semigroups')

We consider the nonhomeogeneous Cauchy problem

$$\frac{d}{dt}u(t) = Au(t) + f(t),\ t > t_0; \quad u(t_0) = x_0, \tag{1.1}$$

in the following situation:

Assumptions 1.1.

a) A is a linear closed operator on a Banach space X such that $\lambda - A$ has a bounded inverse for $\lambda > \omega$ and

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}$$

for all $n \in \mathbf{N}, \lambda > \omega$. M, ω are appropriate constants.

b) $x_0 \in X_0 := \overline{D(A)}$.

c) $f : [0, \infty) \to X$ is continuous.

The main results of this section also hold if f is locally Bochner integrable.

If A is a bounded linear operator on X, (1.1) is solved by

$$u(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)f(s)ds \tag{1.2}$$

with $T(t) = e^{At}$ being the uniformly continuous semigroup generated by A.

If A is a densely defined linear operator in X satisying the assumptions 1.1 a) — i.e. A satisfies the conditions of the *Hille&Yosida*-theorem — then we still try formula (1.2). But we cannot solve (1.1) as such because, in general, neither $u(t)$ is differentiable nor an element in $D(A)$. A very convenient way to fix this problem consists in integrating (1.1) in time and interchanging A with the integral.

Definition 1.2. A continuous function $u : [0,\infty) \to X$ is called an *integral solution* to (1.1) iff

$$u(t) = x_0 + A\int_{t_0}^{t} u(s)ds + \int_{t_0}^{t} f(s)ds \tag{1.3}$$

for $t \geq t_0$.

Note that (1.3) implicitly contains that $\int_{t_0}^{t} u(s)ds \in D(A)$.

Da Prato and Sinistrari [D1] show that there still exist solutions to (1.3) if A is not densely defined, but the assumptions 1.1 hold.

Theorem 1.3. *There is a unique continuous solution u to (1.3) with values in $X_0 = \overline{D(A)}$. u satisfies the estimate*

$$\|u(t)\| \leq Me^{\omega(t-t_0)}\|x_0\| + \int_{t_0}^{t} Me^{\omega(t-s)}\|f(s)\|ds,\ t \geq t_0.$$

Da Prato's und Sinestrari's [D1] following result concerning classical solutions of (1.1) is an easy consequence. See [K2], theorem 2.5, too.

Corollary 1.4. *Let f be absolutely continuous, i.e.*

$$f(t) = f(0) + \int_{0}^{t} g(s)ds$$

for some locally Bochner integrable function g. Let $x_0 \in D(A)$ and $Ax + f(t_0) \in X_0$. Then there exist a unique classical solution u to (1.1), i.e., for $t \geq t_0$, $u(t)$ is continuously differentiable, takes values in $D(A)$, and satisfies (1.1).

Actually, by theorem 1.3, there exists a continuous solution v of

$$v(t) = Ax_0 + f(t_0) + A\int_{t_0}^{t} v(s)ds + \int_{t_0}^{t} g(s)ds.$$

Obviously

$$u(t) = x_0 + \int_{t_0}^{t} v(s)ds$$

is a continuously differentiable solution of (1.1).

The estimate in theorem 1.3 suggests that u can be represented by a variation of constants formula similar to (1.2).

Let us introduce the *part* A_0 of A in $X_0 = \overline{D(A)}$:

$$A_0 = A \quad \text{on} \quad D(A_0) = \{x \in D(A); Ax \in X_0\}. \tag{1.4}$$

Proposition 1.5. *The part A_0 of A in X_0 generates a strongly continuous semigroup $T_0(t), t \geq 0$, on X_0.*

This well-known result is the linear special case of the Crandall&Liggett theorem. It is easy to check that A_0 is densely defined in X_0. As A_0 inherits the estimates in assumption 1.1 a), Proposition 1.5 follows from the Hille&Yosida theorem. See, e.g. [P2], theorem 1.5.3.

So formula (1.2) – with T_0 replacing T – still works if f takes values in X_0. But this is too restrictive in many applications as we have seen in the introduction and is further illustrated in part II.

Let us assume for a moment that f does take values in X_0. Then (1.2) can be written as

$$u(t) = T_0(t-t_0)x_0 + \int_{t_0}^{t} T_0(t-s) \lim_{\lambda\to\infty} \lambda(\lambda - A)^{-1} f(s)ds,$$

or,

$$u(t) = T_0(t-t_0)x_0 + \lim_{\lambda\to\infty} \int_{t_0}^{t} T_0(t-s)\lambda(\lambda - A)^{-1} f(s)ds. \tag{1.5}$$

If f takes values in X, but not in X_0, the limit in the right hand side of the first equations does not longer exist, while, as one of the magics of integration, the limit in the right hand side of (1.5) still exists (as we will prove). We first show that a solution of (1.3) with values in X_0 is actually represented by (1.5):

Let

$$u_\lambda(t) = \lambda(\lambda - A)^{-1}u(t), \quad f_\lambda(t) = \lambda(\lambda - A)^{-1}f(t), \quad x_\lambda = \lambda(\lambda - A)^{-1}x_0.$$

Then, by applying $\lambda(\lambda - A)^{-1}$ to (1.3) and using the smoothening properties of the resolvent,

$$\frac{d}{dt}u_\lambda(t) = A_0 u_\lambda(t) + f_\lambda(t), \quad u_\lambda(t_0) = x_\lambda, \tag{1.6}$$

with $\frac{d}{dt}u_\lambda(t), A_0u_\lambda(t)$ being continuous. Hence, by standard semigroup theory — e.g. [P2], corollary 4.2.2 —

$$u_\lambda(t) = T_0(t-t_0)x_\lambda + \int_{t_0}^{t} T(t-s)f_\lambda(s)ds.$$

As $u(t), x_0 \in X_0$, we have

$$u_\lambda(t) \to u(t), \quad T_0(t-t_0)x_\lambda \to T_0(t-t_0)x_0$$

and (1.5) follows. So we have proved the following variation of constants formula for solutions to (1.3).

Corollary 1.6. *The unique continuous solutions to (1.3) with values in X_0 are given by (1.5).*

Since all the theory of this paper is based on theorem 1.3 and corollary 1.6 we will give an alternative proof of theorem 1.3 by proving corollary 1.6 directly. The proof will be built on proposition 1.5 (i.e. the Hille&Yosida theorem essentially) and on ideas from the theory of 'integrated semigroups' — see [A3, A4, A5, K1, K2, T1] — in particular from [K1], II, theorem 4.3, and [K2], theorem 2.5, and from the theory of weakly* continuous semigroups on dual Banach spaces — see [C1, C2, C3, C4, C5] — without going into these theories more deeply than necessary.

The main idea consists in looking at a special case of the problem, namely $x_0 = 0$ and $f(t) \equiv x, x \in X$.

We define

$$S(t)x = \lim_{\lambda\to\infty} \int_0^t T_0(s)\lambda(\lambda - A)^{-1}xds \tag{1.7}$$

for $x \in X, t \geq 0$.

This definition needs justification.

Lemma 1.7. *For $x \in X, t \geq 0$, the limit in (1.7) exists and defines a bounded linear operator $S(t)$.*

Proof: Let

$$S_0(t)x_0 = \int_0^t T_0(s)x_0ds \tag{1.8}$$

for $t \geq 0, x_0 \in X_0$. Then the definition

$$S(t) = (\lambda - A) S_0(t) (\lambda - A)^{-1} \tag{1.9}$$

for $\lambda > \omega$ extends $S_0(t)$ from X_0 to X. The definition is independent of λ due to the resolvent identity. As $S(t)$ maps X into X_0, we have

$$S(t)x = \lim_{\lambda \to \infty} \lambda(\lambda - A)^{-1} S(t)x = \lim_{\lambda \to \infty} S_0(t) \lambda(\lambda - A)^{-1} x.$$

One can show that $S(t), t \geq 0$, is the 'integrated semigroup' generated by A. See the remarks at the end of this section. In the sequel we only need the following relations.

Lemma 1.8. a) *For $x \in X, t \geq 0$, $\int_0^t S(r)x dr \in D(A)$ and*

$$A \int_0^t S(r)x dr = S(t)x - tx.$$

b) *For $x \in D(A)$, $T_0(t)x = x + S(t)Ax$.*

Proof: a) follows easily from lemma 1.7, the closedness of A and the fact that

$$A \int_0^t T_0(s) x_0 ds = T_0(t)x_0 - x_0$$

for $x_0 \in X_0$.

b) By lemma 1.7,

$$S(t)Ax = \lim_{\lambda \to \infty} \int_0^t T_0(s) \lambda(\lambda - A)^{-1} Ax ds$$

$$= \lim_{\lambda \to \infty} \int_0^t T_0(s) A_0 \lambda(\lambda - A)^{-1} x ds = \lim_{\lambda \to \infty} T_0(t) \lambda(\lambda - A)^{-1} x - \lim_{\lambda \to \infty} \lambda(\lambda - A)^{-1} x = T_0(t)x - x.$$

We are now prepared for the

Proof of theorem 1.3: Actually it is sufficient to proof the theorem for $x_0 = 0$ because the theorem is easily proved for the special case $f \equiv 0$.

Step1: Assume that f is continuously differentiable. Then

$$u_\lambda(t) := \int_{t_0}^t T_0(t - s) \lambda(\lambda - A)^{-1} f(s) ds$$

$$= \int_0^{t-t_0} T_0(s)ds\lambda(\lambda - A)^{-1} f(t_0) + \int_{t_0}^{t} \left(\int_0^{t-r} T_0(s)ds \right) \lambda(\lambda - A)^{-1} f'(r)dr.$$

By lemma 1.7 the limit

$$u(t) = \lim_{\lambda\to\infty} u_\lambda(t)$$

exists and, by using lemma 1.7 and 1.8 several times and the closedness of A, we obtain that

$$u(t) = S(t - t_0)f(t_0) + \int_{t_0}^{t} S(t - r)f'(r)dr$$

$$= A\int_0^{t-t_0} S(r)drf(t_0) + (t - t_0)f(t_0) + \int_{t_0}^{t} \left(A \left(\int_0^{t-r} S(s)ds \right) f'(r) + (t - r)f'(r) \right) dr$$

$$= A\left(\int_0^{t-t_0} S(r)drf(t_0) + + \int_{t_0}^{t} (\int_0^{t-r} S(s)ds)f'(r)dr \right) + (t-t_0)f(t_0) + \int_{t_0}^{t} (t-r)f'(r))dr$$

$$= A\left(\int_{t_0}^{t} u(r)dr \right) + \int_{t_0}^{t} f(r)dr.$$

Step 2: Approximation by continuously differentiable functions

We approximate f by continuously differentiable functions f_n such that

$$\int_{t_0}^{t} \|f(r) - f_n(r)\|dr \to 0$$

for $n \to \infty, t > t_0$. Let u_n be given by (1.5) with f_n replacing f . It follows from (1.5) that the u_n form a Cauchy sequence in the topology of locally uniform convergence. We have seen in step 1 that they satisfy (1.3). As A is a closed operator, the limit u satisfies (1.3), too. As we have shown in corollary 1.4, u is given by (1.5).

The estimate in theorem 1.3 follows from (1.5). If done directly, the estimate follows with M^2 instead of M. This can be avoided by taking an equivalent norm in which the estimates in assumption 1.1 a) and proposition 1.5. hold with $M = 1$. See [P2], lemma 1.5.1. Going back to the original norm gives the factor M.

We add a regularity result which is in between theorem 1.3 and corollary 1.4.

Theorem 1.9. *Let the assumptions 1.1 be satisfied and u be the continuous solution to (1.3). Then*

$$\frac{d^+}{dt}u(t_0) := \lim_{h\downarrow 0} \frac{1}{h}[u(t_0 + h) - u(t_0)]$$

exists iff

$$x_0 \in D(A),\ Ax_0 + f(t_0) \in X_0.$$

Moreover, if one (and then both) of these conditions are satisfied, we have

$$\frac{d^+}{dt} u(t_0) = Ax_0 + f(t_0).$$

Proof: Only if follows from (1.3) and the closedness of A.
If: u is given by formula (1.5) which can be rewritten in the form

$$u(t) = T_0(t-t_0)x_0 + \lim_{\lambda\to\infty} \int_{t_0}^{t} T_0(t-s)\lambda(\lambda - A)^{-1}[f(s) - f(t_0)]ds + S(t-t_0)f(t_0).$$

See lemma 1.7. It follows from the continuity of f that

$$\frac{1}{h} \lim_{\lambda\to\infty} \int_{t_0}^{t_0+h} T_0(t_0+h-s)\lambda(\lambda-A)^{-1}[f(s)-f(t_0)]ds \to 0 \quad \text{for} \quad h \to 0.$$

Hence

$$\frac{1}{h}[u(t_0+h) - T_0(h)x - S(h)f(t_0)] \to 0 \quad \text{for} \quad h \to 0.$$

If $x_0 \in D(A)$ we have

$$T_0(h)x_0 - x_0 + S(h)f(t_0) = S(h)(Ax_0 + f(t_0))$$

by lemma 1.8 b). Hence, if $Ax + f(0) \in X_0$, it follows that

$$\frac{1}{h}[u(t_0+h) - x_0] = \frac{1}{h}\int_0^h T_0(s)ds(Ax_0 + f(t_0)) \to Ax_0 + f(t_0)$$

for $h \to 0$.

Remarks 1.10. a) Using $S(t)$ defined in (1.7), (1.8), and (1.9), one can derive the following alternative variation of constants formula for solutions to (1.3):

$$u(t) = T_0(t-t_0)x_0 + \frac{d}{dt}\int_{t_0}^{t} S(t-s)f(s)ds.$$

This can easily been seen by integrating the last term in (1.5) and using lemma 1.7. Formula (1.5) will be more useful for our purposes, however.

b) Though we prefer formula (1.5), some knowledge about $S(t)$ can be useful in studying the invariance of closed convex sets. See theorem 2.11. One readily derives from (1.7) and (1.8) that

$$S(t)S(r) = \int_0^t (S(\tau + r) - S(\tau))d\tau, \quad t, r \geq 0,$$

$$S(0) = 0.$$

Moreover, from (1.7),

$$\|S(t) - S(r)\| \leq Me^{\omega t}(t - r), \quad t \geq r \geq 0.$$

See the preceeding remarks as for why one obtains the estimate with M instead of M^2.

Hence S is a locally Lipschitz norm-continous 'integrated semigroup'. See [A3, A4, K1, K2]. Further one easily realizes from (1.7), (1.8) that, for $\lambda > \omega$, $\lambda - A$ is invertible and

$$(1.10) \qquad (\lambda - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt.$$

Finally one can show the following converse of lemma 1.8 b):

If $x \in X_0, y \in X$ and $T_0(t)x = x + S(t)y$ for all $t \geq 0$, then $x \in D(A)$ and $y = Ax$. This justifies to call A the generator of A. See [T1], definition 3.1 and proposition 3.10. Note that A uniquely determines S by (1.10) as well by being its generator. The first follows from the uniqueness of the Laplace transform, the second from [T1], theorem 3.6. (1.10) can often be used in order to determine S explicitly.

c) It is worth mentioning that one can also take the opposite route, i.e. start from an 'integrated semigroup' instead of starting from an operator. The generator is then defined by

$$x \in D(A), y = Ax \iff \frac{d}{dt} S(t)x = x + S(t)y, t \geq 0.$$

If the 'integrated semigroup' is exponentially bounded, one can also use (1.10) in order to define A. It turns out that S is locally Lipschitz norm-continuous if and only if A satisfies the estimates in assumption 1.1 a. See [K2], theorem 2.4.

d) Finally we remark that the solutions u to (1.3) can be expressed by a variation of constants formula of form (1.2) but not with a strongly continuous semigroup T acting on X, but on a larger space X_λ with a weaker norm. See [C5], section 5, and [T1], section 5. Actually X_λ is the completion of X under one of the equivalent norms $\|x\|_\lambda = \|(\lambda - A)^{-1}x\|$, $\lambda > \omega$. T is the continuous extension of T_0 from X_0 to X_λ. The infinitesimal generator A_λ of T is the closure of A in X_λ and A is the part of A_λ in X. Eventually $D(A_\lambda) = X_0$ and

$$\frac{d}{dt}u(t) = A_\lambda u(t) + f(t),\ t > 0$$

holds in X_λ. This follows from (1.6): First one realizes that u is differentiable in X_λ. Using that $A_0(\lambda - A)^{-1} = (\lambda - A_\lambda)^{-1}A_\lambda$ on X_0 yields the differential equation. Compare [A1], remark 6.8.

2.1 Solutions to abstract Cauchy problems which leave invariant a closed convex set

We consider the abstract semilinear Cauchy problem

$$(\diamondsuit) \qquad \frac{d}{dt}u(t) = Au(t) + F(t)u(t),\ t > t_0 \geq 0; \quad u(t_0) = x_0,$$

in the following situation:

Assumptions 2.1. a) *A is a closed linear operator on a Banach space X. $(\lambda - A)$ has a bounded linear inverse on X and*

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}$$

for all $n \in \mathbf{N}, \lambda > \omega$ with appropriate real constants M, ω.

b) *A is not necessarily densely defined, however. Let $X_0 = \overline{D(A)}$. We assume that the initial value x_0 in $(\diamondsuit)$ is an element in X_0, and we look for solutions u of $(\diamondsuit)$ with values in X_0. Actually we are interested in finding a solution u with values in*

$$C_0 = C \cap X_0$$

with C being a closed convex subset of the Banach space X. Note that C_0 is a closed convex subset,too. So we assume that

$$(2.1) \qquad x_0 \in C_0.$$

c) *We assume the following properties of the operators*

$$F(t) : C_0 \to X.$$

(i) For any $x \in C_0$, $F(t)x$ is a continuous function of $t \geq 0$.

(ii) For any $t \geq 0, x \in C_0$ there exist $\delta, \Lambda > 0$ such that

$$\|F(s)y - F(s)z\| \leq \Lambda \|y - z\|, \tag{2.2}$$

if $t \leq s \leq t + \delta$, $y, z \in C_0$, $\|y - x\|, \|z - x\| \leq \delta$.

(iii) For any $\tau > 0$ there exists some $c > 0$ such that

$$\|F(t)x\| \leq c(1 + \|x\|)$$

if $0 \leq t \leq \tau$, $x \in X_0$.

Remarks: This problem has the special feature that A is not densely defined in X, but that the nonlinearities $F(t)$ may be only defined on a subset of $X_0 = \overline{D(A)}$ and map into X, but out of X_0.

The assumptions above are not yet sufficient to guarantee that the solution is going to stay in C_0. To this end we assume that C is invariant under $\lambda(\lambda - A)^{-1}$ and that F satisfies a *subtangential condition.* Compare [M1], VIII, and [M2].

Assumptions 2.2. a) *$\lambda(\lambda - A)^{-1}$ maps C into itself for sufficiently large $\lambda > \omega$.*
b) *For $t \geq 0, y \in C_0$*

$$\frac{1}{h} dist(y + hF(t)y; C) \to 0 \quad \text{as} \quad h \downarrow 0.$$

Here

$$dist(z; C) = \inf_{x \in C} \|z - x\|$$

gives the distance of a point $z \in X$ from the set C.

In general we cannot solve ($\diamondsuit$) in this strong formulation, if $x \in C_0 \setminus D(A)$. So, for arbitrary $x \in C_0$, we solve it in the integrated form

$$u(t) = x_0 + A \int_{t_0}^{t} u(s)ds + \int_{t_0}^{t} F(s)u(s)ds, \; t \geq t_0. \tag{$\heartsuit$}$$

A solution to ($\heartsuit$) is called an *integral solution* to ($\diamondsuit$).

Theorem 2.3. *Let the assumptions 2.1 and 2.2 be satisfied. Then there exists a unique continuous solution to* (♡) *with values in* C_0*.*

The rest of this section is devoted to the proof of theorem 2.3. A more general version of theorem 2.3 is presented in theorem 2.11.

We use the variation of constants formula (1.5). See theorem 1.3 and corollary 1.5. Recall that the part A_0 of A in X_0 generates a strongly continuous linear semigroup T_0 on X_0. Now (♡) can be equivalently written in the form

$$(\spadesuit) \qquad u(t) = T_0(t-t_0)x_0 + \lim_{\lambda\to\infty}\int_{t_0}^{t} T_0(t-s)\lambda(\lambda-A)^{-1}F(s)u(s)ds, \quad t \geq t_0.$$

It is convenient to renormalize X in an equivalent way such that

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda-\omega} \tag{2.3}$$

for $\lambda > \omega$. See [P2], lemma 1.1.5. This renormalization has also the consequence that

$$\|T_0(t)\| \leq e^{\omega t} \tag{2.4}$$

for $t \geq 0$.

We collect some properties of *dist*.

Lemma 2.4. *Let D, D_0 be closed subsets of X and $x, y \in X$. Then the following holds:*

a)

$$|dist(x;D) - dist(y;D)| \leq \|x-y\|.$$

b) $x \in D$ iff $dist(x;D) = 0$.

c) *If B is a linear bounded operator on X which maps D into D_0, then*

$$dist(Bx;D_0) \leq \|B\| dist(x;D).$$

These properties follow easily from the definition of *dist*.

Lemma 2.5. *For $x \in X_0$, $dist(x;C) = dist(x;C_0)$.*

Proof: Obviously $dist(x; C) \leq dist(x; C_0)$, because $C_0 \subset C$.
As $x \in C_0$, $dist(x; C_0) = \lim_{\lambda \to \infty} dist(\lambda(\lambda - A)^{-1})x; C_0) \leq dist(x; C)$. Here we have used lemma 2.4 a), c), and (2.3).

Lemma 2.6. a) *$dist(x; C_0)$ is a convex function of $x \in X$.*

b) *$\frac{1}{h}\big(dist(x + hy; C_0) - dist(x; C_0)\big)$ is a monotone non-increasing function of $h > 0$.*

Proof: a) follows from the convexity of C_0. b) follows from a).

Lemma 2.7. *(Jensen's inequality) Let u, ϕ be continuous functions on $[t_1, t_2]$ with values in $X, [0, \infty)$, respectively. Let $\int_{t_1}^{t_2} \phi(s)ds = 1$. Let ψ be a convex real-valued function on X. Then*

$$\psi(\int_{t_1}^{t_2} \phi(s)u(s)ds) \leq \int_{t_1}^{t_2} \phi(s)\psi(u(x))ds.$$

Lemma 2.8. *$T_0(t)$ maps C_0 into C_0.*

Proof: Recall that, for $x \in X_0$,

$$T_0(t)x = \lim_{n \to \infty} (I - \frac{t}{n}A)^{-n}x$$

and apply assumption 2.2 a).

Remark: Actually the invariance of C under $\lambda(\lambda - A)^{-1}$ for large λ — i.e. assumption 2.2 a) — is equivalent to the invariance of C under $\frac{1}{t}S(t)$ for all $t > 0$. Here S is the 'integrated semigroup' generated by A.
This can be seen from (1.7), (1.10) and lemma 2.7, 2.8.

The following lemma will provide us with the tool to construct the solution of (♠). The proof of this lemma follows the lines of [M2], corollary 1, but requires some extra effort due to the fact that T_0 only operates on X_0 and $F(t)$ maps out of X_0.

Lemma 2.9. *For any $x \in C_0, t \geq 0$,*

$$\frac{1}{h}dist\left(T_0(h)x + \lim_{\lambda \to \infty} \int_0^h T_0(s)\lambda(\lambda - A)^{-1}F(t)xds; C_0\right) \to 0$$

for $h \downarrow 0$.

Proof: The above expression can be estimated by

$$\leq \frac{1}{h} dist\left(\int_0^1 T_0(sh)T_0([1-s]h)xds + \lim_{\lambda\to\infty}\int_0^1 T_0(hs)\lambda(\lambda-A)^{-1}hF(t)xds; C_0\right)$$

$$\leq \frac{1}{h} dist\left(\lim_{\lambda\to\infty}\int_0^1 T_0(hs)\lambda(\lambda-A)^{-1}\Big(T_0([1-s]h)x + hF(t)x\Big)ds; C_0\right).$$

By lemma 2.4 a) we can continue by

$$\leq \frac{1}{h}\lim_{\lambda\to\infty} dist\Big(\int_0^1 T_0(hs)\lambda(\lambda-A)^{-1}\big(T_0([1-s]h)x + hF(t)x\big)ds; C_0\Big).$$

By lemma 2.4 c), Jensen's inequality (lemma 2.7), and (2.3), (2.4) we can continue by

$$\leq \frac{1}{h}\int_0^1 e^{\omega hs} dist\big(T_0([1-s]h)x + hF(t)x; C\big)ds.$$

By the continuity of F we can continue by

$$\leq \frac{1}{h}\int_0^1 e^{\omega hs} dist\Big(T_0([1-s]h)x + hF(t)T_0([1-s]h)x; C\Big)ds + \eta(h).$$

with $\eta(h) \to 0$ for $h \downarrow 0$.
By lemma 2.6 b) and Dini's lemma we obtain from assumption 2.2 b) that

$$\frac{1}{h} dist\big(T_0(r)x + hF(t)T_0(r)x; C\big) \to 0$$

for $h \downarrow 0$ uniformly in $r \in [0,1]$, because $\{T_0(r)x; 0 \leq r \leq 1\}$ forms a compact subset of C_0.

This implies the assertion.

The idea in proving the existence of solutions to (♡) consists in first proving the unique existence of local solutions to (♠).

Proposition 2.10. *Let the assumptions 2.1 and 2.2 be satisfied. Then there exists a unique continuous solution u to (♡) on an interval $[t_0, t_0+\tau]$ with values in C_0. τ may depend on x_0.*

The proof of proposition 2.10 only relies on lemma 2.9 and not on the assumptions 2.2. It is essentially the same as in [M2], section 5, (see also [M1], chapter 8) and is given in the appendix for completeness. Note that [M2] replaces the convexity of C by a more general condition.

Now we can give the proof of theorem 2.3 by a maximum extension argument.

Proof of theorem 2.3: Let τ be the greatest positive number such that we can find unique continuous solutions to (♡) on $[t_0, t_0+\sigma]$ for any $\sigma < \tau$. By uniqueness we have a solution u on $[t_0, t_0+\tau)$. We have to show that $\tau = \infty$. So let us suppose that τ is finite. By assumption 2.1 (iii) we find some $c > 0$ such that

$$\|u(t)\| \leq e^{\omega t}\|x_0\| + c\int_{t_0}^{t} e^{\omega(t-s)}[1 + \|u(s)\|]ds$$

if $t_0 \leq t \leq t_0+\tau$. A Gronwall type argument implies that u is bounded on $[t_0, t_0+\tau]$. Thus u can be continuously extended to a solution of (♡) on $[t_0, t_0+\tau]$ by

$$u(t_0+\tau) = T_0(\tau)x_0 + \lim_{\lambda\to\infty}\int_{t_0}^{t_0+\tau} T_0(t_0+\tau-s)\lambda(\lambda-A)^{-1}F(s)u(s)ds$$

$$= T_0(\tau)x_0 + \lim_{\lambda\to\infty}\int_{0}^{\tau} T_0(s)\lambda(\lambda-A)^{-1}F(t_0+\tau-s)u(t_0+\tau-s)ds.$$

By proposition 2.12 we find a unique continuous solution

$$u(t) = u(t_0+\tau) + A\int_{t_0+\tau}^{t} u(s)ds + \int_{t_0+\tau}^{t} F(s)u(s)ds$$

for $t_0+\tau \leq t \leq t_0+\tau+\delta$ with some sufficiently small $\delta > 0$. Using the fact that u solves (♡) on $[t_0, t_0+\tau]$ and manipulating the integrals we easily realize that we have obtained a unique solution of (♡) on $[t_0, t_0+\tau+\delta]$. This contradicts the maximality of τ.

Assumption 2.2 has been used in the proof of theorem 2.3 only to obtain lemma 2.9. Note from lemma 1.7 that the statement of lemma 2.9 can be rewritten in a condensed form by using the 'integrated semigroup' S generated by A. The statement of lemma 2.9 is even necessary to obtain a solution of (♡) with values in C_0.

Theorem 2.11. *Let the assumptions 2.1 be satisfied. Then there exists a continuous solution to (♡) with values in C_0 for any initial condition $x_0 \in C_0$ at time $t_0 \geq 0$ iff*

$$\frac{1}{h}dist\Big(T_0(h)x + S(h)F(t)x; C_0\Big) \to 0$$

for $h \downarrow 0$, $t \geq t_0, x \in C_0$. Here S denotes the 'integrated semigroup' generated by A. The solution (if it exists) is unique.

Proof: We still have to show that the above subtangential condition is necessary.

Let $t \geq t_0, x \in C_0$. Then there exists a solution u of (♡) with initial condition x at time t. u has values in C_0 and is given by

$$u(t+h) = T_0(h)x + \lim_{\lambda\to\infty} \int_t^{t+h} T_0(t+h-s)\lambda(\lambda - A)^{-1} F(s)u(s)ds, \quad h \geq 0.$$

Hence

$$\frac{1}{h} dist\big(T_0(h)x + S(h)F(t)x; C_0\big)$$

$$\leq \frac{1}{h} \|T_0(h)x + S(h)F(t)x - u(t+h)\|$$

$$\leq \frac{1}{h} \| \lim_{\lambda\to\infty} \int_t^{t+h} T_0(t+h-s)\lambda(\lambda - A)^{-1} \Big(F(t)u(t) - F(s)u(s)\Big) ds\|$$

$$\leq \frac{1}{h} \int_t^{t+h} e^{\omega(t+h-s)} \|F(t)u(t) - F(s)u(s)\| ds.$$

The assertion now follows from the continuity of $F(s)u(s)$.

3. The semiflow and its properties

The unique solutions u of

$$(\heartsuit) \qquad u(t) = x_0 + A\int_{t_0}^t u(s)ds + \int_{t_0}^t F(s)u(s)ds, \; t \geq t_0$$

which we found in the previous section induce a semiflow (nonlinear evolutionary system) U on the closed convex set C_0 by setting

$$U(t, t_0)x_0 = u(t).$$

Actually, by manipulating the integrals in (♡) we easily see that

(3.1) $$U(t,s)U(s,r) = U(t,r), \quad t \geq s \geq r \geq 0.$$

$$U(s,s)x = x, \quad s \geq 0, x \in C_0.$$

We study the properties of the semiflow U. In order to avoid technicalities we strengthen the Lipschitz condition for F. So we assume the following from now on.

Assumptions 3.1. a) *A is a closed linear operator on a Banach space X. A satisfies the Hille&Yosida estimates, i.e. $(\lambda - A)$ has a bounded linear inverse on X and*

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}$$

for all $n \in \mathbf{N}, \lambda > \omega$ with appropriate real constants M, ω.

b) *We assume the the following properties of the operators*

$$F(t) : C_0 \to X.$$

(i) For any $x \in C_0$, $F(t)x$ is a continuous function of $t \geq 0$.
(ii) There exists $\Lambda > 0$ such that

$$\|F(s)y - F(s)z\| \leq \Lambda \|y - z\|,$$

for all $s \geq 0,\ y, z \in C_0$.

c) *$\lambda(\lambda - A)^{-1}$ maps C into itself for sufficiently large $\lambda > \omega$.*

d) *For $t \geq 0, y \in C_0$*

$$\frac{1}{h} dist(y + hF(t)y; C) \to 0 \quad \text{as} \quad h \downarrow 0.$$

We recall that assumption c) and d) can be replaced by the more general subtangential condition

$$\frac{1}{h} dist\Big(T_0(h)x + S(h)F(t)x; C_0\Big) \to 0$$

for $h \downarrow 0$, $t \geq 0, x \in C_0$. Here S denotes the 'integrated semigroup' generated by A. See theorem 2.11. The global Lipschitz condition b) (ii) can be replaced by a linear growth condition and a Lipschitz condition on bounded subsets of C_0 . The estimates in the following theorems then hold on bounded subsets of C_0 only.

First we investigate the continuity properties of U.

Theorem 3.2. **a)**$U(t,s)$ *is a Lipschitz continuous operator. More precisely*

$$\|U(t,s)x - U(t,s)y\| \leq M\|x-y\|e^{(\omega+M\Lambda)(t-s)}$$

for $t \geq s \geq 0,\ x,y \in C_0$.

b) $U(t,s)x$ *is a continuous function of* (t,s,x).

Proof: a) Let $u(t) = U(t,s)x,\ v(t) = U(t,s)y,\ w = u - v$. Then

$$w(t) = x - y + A\int_s^t w(r)dr + \int_s^t \big(F(r)u(r) - F(r)v(r)\big)dr.$$

By theorem 1.3

$$\|w(t)\| \leq Me^{\omega(t-s)} + M\Lambda\int_s^t e^{\omega(t-r)}\|w(r)\|dr.$$

Gronwall's inequality now provides the assertion.

b) *Step 1:* We already know that $U(t,s)x$ is continuous as a function of t.

Step 2: $U(t,r)x \to x, \quad t,r \to s,\ t \geq r.$

Indeed, from (♠) in section 2,

$$\|U(t,r) - x\| \leq \|T_0(t-r)x - x\| + M\int_r^t e^{\omega(t-\sigma)}\|F(\sigma)u(\sigma)\|d\sigma.$$

Step 3: $U(t,s)x$ is continuous in s locally uniformly in t.

Let $s > r$. Then

$$\|U(t,s)x - U(t,r)x\| \leq \|U(t,s)x - U(t,s)U(s,r)x\|.$$

The statement now follows from the Lipschitz continuity of $U(t,s)$. See part a).

Taking these steps together yields b).

Similarly as for linear evolutionary systems we can define the infinitesimal generators of U:

$$G_0(t)x = \lim_{h\downarrow 0}\frac{1}{h}\big(U(t+h,t)x - x\big)$$

with the domain of $G_0(t)$ consisting of those elements $x \in C_0$ for which the limit exists.

Theorem 1.9 provides the following characterisation of G_0.

Theorem 3.3. *$G_0(t)$ is the part of $A + F(t)$ in X_0, i.e.*

$$G_0(t) = A + F(t) \quad \text{on} \quad D(G_0(t)) = \{x \in C_0 \cap D(A); Ax + F(t)x \in X_0\}.$$

For studying the stability of steady states and other states of the semiflow, differentiability of $U(t,s)x$ in x is paramount.

Theorem 3.4. *Let $F(t) : C_0 \to X$ be continuously differentiable. More precisely assume that, for any $x \in C_0, t \geq 0$, there exists a bounded linear operator $\partial_x F(t,x) = \partial_2 F(t,x)$ from X_0 to X such that*

$$\frac{1}{\|z\|}\Big(F(t)(x+z) - F(t)x - \partial_x F(t,x)z\Big) \to 0$$

for $0 \neq z \to 0, x + z \in C_0$. Further let $\partial_x F(t,x)z$ be a continuous function of (t,x) for any $z \in X_0$ and $\partial_x F(t,x)$ be a continuous function of x from C_0 into the bounded linear operators with the uniform operator topology.

Then $U(t,r)x$ is differentiable in $x \in C_0$ and $\partial_x U(t,r)x$ is the linear evolutionary system generated by the solutions $w(t) = (\partial_x U(t,r)x)z$ of the problem

$$w(t) = z + A\int_r^t w(s)ds + \int_r^t \partial_2 F(s, U(s,r)x)w(s)ds, \ t \geq r.$$

Further, for any $z \in X_0$, $(\partial_x U(t,r)x)z$ is continuous in (t,r,x).

Proof: Let $u(t) = U(t,r)x, v(t) = U(t,r)y$, and w as in the statement of the theorem with $z = x - y$. Unique existence of w and the continuous dependence on (t,r) of the associated linear evolutionary system follow from theorem 3.2. Continuous dependence on x needs another application of theorem 1.3 and Gronwall's inequality which is left to the reader. Note that $\|\partial_x F(t,x)\| \leq \Lambda$ by the Lipschitz condition in assumption 3.1 b) (ii). For proving the differentiability of $U(t,r)x$ in x we set $q = v - u - w$. Then

$$q(t) = A\int_r^t q(s)ds + \int_r^t \Big(F(s)v(s) - F(s)u(s) - \partial_x F(s,u(s))w(s)\Big)ds$$

$$= A\int_r^t q(s)ds + \int_r^t \partial_x F(s,u(s))q(s)ds$$

$$+ \int_r^t \int_0^1 \Big(\partial_x F\Big(s, u(s) + \xi(v(s) - u(s))\Big) - \partial_x F(s,u(s))\Big)d\xi \ \big(v(s) - u(s)\big)ds.$$

Hence, by theorem 1.3,

$$\|q(t)\| \leq M\Lambda \int_r^t e^{\omega(t-s)} \|q(s)\| ds + \phi(t, r, x, y)$$

with

$$\frac{\phi(t, r, x, y)}{\|x - y\|} \to 0$$

if $y \to x$. Recall that $\|v(s) - u(s)\| \leq const\|y - x\|$ on every bounded interval by proposition 3.2. Differentiability of $U(t, r)x$ in x now follows from Gronwall's inequality.

Corollary 3.5. *Let F satisfy the assumptions of theorem 3.4 and $x \in D(G_0(s))$, i.e. $x \in D(A), Ax + F(s)x \in X_0$. Then $U(t, s)x$ is continuously differentiable in s and*

$$\partial_s U(t, s)x = -(\partial_x U(t, s)x)(Ax + F(s)x).$$

Proof: We prove differentiability from the right. This is sufficient because the right derivative turns out to be continuous. Now

$$U(t, s)x - U(t, s + h)x = U(t, s + h)U(s + h)x - U(t, s + h)x$$

$$= \int_0^1 \partial_x U(t, s + h)\big(x + \xi(U(s + h, s)x - x)\big)d\xi \; (U(s + h, s)x - x).$$

The assertion follows from theorem 3.3 and 3.4.

Before we study the differentiability of $U(t, r)x$ in t we deal with the following question.

Proposition 3.6. *Let F satisfy the assumptions of theorem 3.4 and $F(t)x$ be differentiable in t for any $x \in C_0$ with $\partial_t F(t)x = \partial_1 F(t, x)$ being a continuous function of (t, x). Then the directional derivative $\partial_\sigma U(t + \sigma, r + \sigma)x$ exists and is given by the solutions w of the equation*

$$w(t) = A \int_r^t w(s)ds + \int_r^t \partial_2 F(s + \sigma, U(s + \sigma, r)x)w(s)ds + \int_r^t \partial_1 F(s + \sigma, U(s + \sigma, r)x)ds.$$

In particular $\partial_\sigma U(t + \sigma, r + \sigma)x$ is a continuous function of (t, σ, x).

Proof: Let $w_h(t) = U(t + \sigma + h, r + \sigma + h)x - U(t + \sigma, r + \sigma)x$. Then

$$w_h(t) = A \int_r^t w_h(s)ds + \int_r^t \Big(F(s+\sigma+h)\big(U(s+\sigma, r+\sigma)x + w_h(s)\big) - F(s+\sigma)U(s+\sigma, r)x\Big)ds.$$

The assertion now follows in a similar way as in the proof of proposition 3.4.

We conclude this section by showing that $(\diamondsuit)$ can be solved in the strong sense if we assume enough regularity.

Theorem 3.7. *Let $F(t,x)$ be continuously differentiable both in t and x in the sense of theorem 3.4 and proposition 3.6. Let $x_0 \in D(G_0(t_0))$, i.e. $x \in D(A), Ax + F(t_0)x \in X_0$. Then $u(t) = U(t,t_0)x_0$ has values in $D(A)$, is differentiable in $t \geq t_0$, and*

$$\frac{d}{dt}u(t) = (A + F(t))u(t) = G_0(t)u(t),\ t \geq t_0; \quad u(t_0) = x_0.$$

Proof: It is sufficient to show that $u(t)$ is differentiable Then $f(t) = F(t)u(t)$ is differentiable and the assertion follows from corollary 1.4. It will be sufficient to show differentiability from the right because the right derivative will turn out to be continuous. Let $x \in D(A), Ax + F(t_0)x \in X_0, h > 0$. Then, by theorem 3.4 and proposition 3.6,

$$U(t+h,t_0)x - U(t,t_0)x$$

$$= U(t+h,t_0+h)U(t_0+h,t_0)x - U(t,t_0)U(t_0+h,t_0)x + U(t,t_0)U(t_0+h,t_0)x - U(t,t_0)x$$

$$= \int_0^h \partial_r U(t+r,t_0+r)U(t_0+h,t_0)x dr$$

$$+ \int_0^1 \partial_x U(t,t_0)\Big((1-\xi)x + \xi U(t_0+h,t_0)x\Big)d\xi\Big(U(t_0+h,t_0)x - x\Big).$$

Hence, by theorem 3.3 and 3.4 and by proposition 3.6,

$$\frac{1}{h}\Big(U(t+h,t_0)x - U(t,t_0)x\Big) \to \partial_r U(t+r,t_0+r)\mid_{r=0} x + (\partial_x U(t,t_0)x)(Ax + F(t_0)x).$$

4. Steady states and their stability

We consider the time-homogeneous special case of $(\heartsuit)$, i.e. the Lipschitz perturbations $F(t)$ are independent of t:

$$(\heartsuit) \qquad u(t) = x_0 + A\int_{t_0}^t u(s)ds + \int_{t_0}^t Fu(s)ds,\ t \geq t_0.$$

We again assume the assumptions 3.1. Assumption b) (i) is now redundant. The semiflow U generated by the unique solution u of (♡) becomes time-homogeneous, i.e.

$$U(t,t_0) = U(t-t_0,0) = T(t-t_0)$$

with T being the nonlinear semigroup provided by the solutions

$$u(t) = x_0 + A\int_0^t u(s)ds + \int_0^t Fu(s)ds,\ t \geq 0.$$

Important candidates for the asymptotic behaviour of solutions to (♡) are steady states (or equilibria), i.e. time-independent solutions, of (♡).

The following relations are trivial, but important.

Theorem 4.1. *The following statements are equivalent for $x_0 \in C_0$:*

(i) $u(t) \equiv x_0$ *is a time-independent solution to* (♡).

(ii) $T(t)x_0 = x_0$ *for* $t \geq 0$.

(iii) $x_0 \in D(A)$ *and*

$$Ax_0 + Fx_0 = 0.$$

An important property of steady states is *locally asymptotic stability*: Trajectories which start sufficiently close to the steady state remain close and return to the steady state when time tends to infinity.

If F is continuously Frechet-differentiable in C_0, the nonlinear semigroup T is Frechet-differentiable and $\partial_x T(t)x_0$ is the strongly continuous linear semigroup generated by the part of $A + F'(x_0)$ in X_0. See theorem 3.4 and theorem 3.3. Let

$$\omega_0\Big(A + F'(x_0)\Big) := \inf_{t>0} \frac{1}{t} \ln \|\partial_x T(t)x_0\| = \lim_{t\to 0} \frac{1}{t} \ln \|\partial_x T(t)x_0\|$$

be the *type* or *growth bound* of $\partial_x T(t)x_0$. Then, along the lines of [D2] (see also [C3]), we can prove the following stability result.

Theorem 4.2. *Let the assumptions 3.1 be satisfied and F be continuously Frechet-differentiable in C_0. Let x_0 be a steady state. Let $\omega_0\Big(A + F'(x_0)\Big) < 0$. Then, for any $\omega > \omega_0\Big(A + F'(x_0)\Big)$, there exist $c, \delta > 0$ such that*

$$\|T(t)x - x_0\| \leq ce^{\omega t}\|x - x_0\|$$

for all $x \in C_0$ *with* $\|x - x_0\| \le \delta$.

In particular x_0 is locally asymptotically stable because we may choose $\omega < 0$.

Proof: Without restricting the generality we may assume that $x_0 = 0$ and $\omega < 0$. We choose some α with

$$\omega_0\Big(A + F'(x_0)\Big) < \alpha < \omega < 0.$$

Then we find $M_\alpha > 0$ such that

$$\|\partial_x T(t)x_0\| \le M_\alpha e^{\alpha t}$$

for $t \ge 0$. We choose $t_0 > 0$ such that

$$e^{-\omega t_0}\|\partial_x T(t_0)x_0\| \le M_\alpha e^{(\alpha-\omega)t_0} \le \frac{1}{2}.$$

There exists $\epsilon > 0$ such that

$$\|T(t_0)x - (\partial_x T(t_0)x_0)x\| \le \frac{1}{2}e^{\omega t_0}\|x\|$$

if $\|x\| \le \epsilon$. Hence

$$e^{-\omega t_0}\|T(t_0)x\| \le e^{-\omega t_0}\|T(t_0)x - (\partial_x T(t_0)x_0)x\| + e^{-\omega t_0}\|(\partial_x T(t_0)x_0)x\| \le \|x\|$$

if $\|x\| \le \epsilon$. In particular, $\|T(t_0)x\| \le \epsilon$. By induction we find

$$e^{-k\omega t_0}\|T(kt_0)x\| \le \|x\|,\ k \in \mathbf{N},$$

if $\|x\| \le \epsilon$. By theorem 3.2 we find $M, \beta > 0$ such that

$$\|T(s)x\| \le Me^{\beta s}\|x\|,\ s \ge 0.$$

Choosing $\delta = \epsilon M^{-1}e^{-\beta t_0}$ we have

$$\|T(s)x\| \le \epsilon$$

if $0 \le s \le t_0, \|x\| \le \delta$.

Let $t = kt_0 + s$ with $k \in \mathbf{N}, 0 \le s < t_0$. Then

$$e^{-\omega t}\|T(t)x\| \le e^{-\omega s}e^{-\omega k t_0}\|T(kt_0)T(s)x\| \le e^{-\omega t_0}Me^{\beta t_0}\|x\|,$$

if $\|x\| \le \delta$. This implies the assertion.

In order to obtain a result which may be more easily applied and which also give a condition for instability of a steady state, we introduce the *essential type*

$$\omega_{ess}\Big(A + F'(x_0)\Big) := \inf_{t>0} \frac{1}{t} \ln \|\partial_x T(t) x_0\|_{ess} = \lim_{t\to 0} \frac{1}{t} \ln \|\partial_x T(t) x_0\|_{ess}.$$

Here $\|\cdot\|_{ess}$ denotes an appropriate measure of noncompactness of an operator.

Corollary 4.3. *Let the assumptions 3.1 be satisfied and F be continuously Frechet-differentiable in C_0. Let x_0 be a steady state. Let $\omega_{ess}\Big(A + F'(x_0)\Big) < 0$.*

a) *If all eigenvalues of $A+F'(x_0)$ have strictly negative real part, then there exist $\omega < 0 < \delta, c$ such that*

$$\|T(t)x - x_0\| \leq ce^{\omega t}\|x - x_0\|$$

for all $x \in C_0$ with $\|x - x_0\| \leq \delta, t \geq 0$.

b) *If at least one eigenvalue of $A + F'(x_0)$ has stricly positive real part, then x_0 is an unstable steady state in the following sense:*
There exist a constant $\epsilon > 0$ and sequences $x_n \to 0$ in C_0, $t_n \to \infty$ $(n \to \infty)$ such that $\|T(t_n)x_n - x_0\| \geq \epsilon$ for all $n \in \mathbf{N}$.

Proof: Part a) follows from theorem 4.2. For $\omega_0\Big(A+F'(x_0)\Big) = \max\{\omega_{ess}\Big(A+F'(x_0)\Big), s\Big(A+F'(x_0)\Big)\}$ with $s\Big(A + F'(x_0)\Big)$ denoting the spectral bound, i.e. the supremum of the real parts of the spectral values, of $A + F'(x_0)$. See [C6], proposition 8.6.

Part b) follows from theorem 2.2 in [D2]. Let Σ denote the set of spectral values with positive real part. As these are normal eigenvalues, Σ is finite and bounded away from the imaginary axis and can so be separated from rest of the spectrum by rectifiable simple closed curve. According to [K1], chapter III, theorem 6.17 there exists a decomposition of X into invariant subspaces X_1 and X_2 such that the restriction of $A + F'(x_0)$ to X_1 has the spectrum Σ and its restriction to X_2 has a spectral bound $\leq 0 < \inf\{Re\lambda; \lambda \in \Sigma\}$ and thus a non-positive type. See [C6] theorem 8.6. But this is the situation of theorem 2.2 in [D2] which implies the result.

[D2] also contains results about the stability of periodic solutions.

Appendix

For completeness we translate the proof in [M2] to our situation and give the

Proof of proposition 2.10:

The solution u to (♠) is constructed by a polygonal approximation. First we choose knots $(t_j, x_j),\ t_0 < t_j \le t_0 + \tau,\ x_j \in C_0$ in an appropriate manner (step 1,2). We find an approximate solution of (♠) by connecting the knots by a polygon (step 3). Finally we show by a Gronwall argument that the polygonal approximate solutions converge towards a solution of (♠) (step 4). The uniqueness of solutions follows from the Lipschitz condition for F and a Gronwall argument.

Step 1: Construction of the knots of the appromimate polygon solution

By assumption 2.1 c) we choose $\rho > 0$ such that

$$\|F(t)x - F(t)y\| \le \Lambda \|x - y\|$$

if $t_0 \le t \le t_0 + \rho,\ x, y \in B_\rho(x_0)$.
It follows that $F(t)x$ is uniformly continuous and bounded in (t, x) on $[t_0, t_0 + \rho] \times B_\rho(x_0)$. Here $B_\rho(x_0)$ denotes the ball with radius ρ and center x_0.

We now construct the knots. Let

$$0 < \epsilon < 1$$

be arbitrary. By lemma 2.9 we can successively choose

$$t_j > t_{j-1} > ... > t_0, \quad x_j, ..., x_1 \in C_0$$

such that
(A.1)
$$\|x_j - T_0(t_j - t_{j-1})x_{j-1} - \lim_{\lambda \to \infty} \int_{t_{j-1}}^{t_j} T_0(t_j - s)\lambda(\lambda - A)^{-1} F(t_{j-1})x_{j-1} ds\| \le \epsilon(t_j - t_{j-1}).$$

Further we choose them such that

$$(A.2) \qquad \|T_0(s)x_{j-1} - x_{j-1}\| \le \epsilon$$

if $0 \le s \le t_j - t_{j-1}$, and

$$(A.3) \qquad t_j - t_{j-1} \le \epsilon,\ t_j \le t_0 + \tau.$$

Choose the t_j as large as possible under the constraints of (A.1),..., (A.3).

We need to guarantee that this construction provides elements $x_j \in B_\rho(x_0)$, as long as $t_j \leq t_0 + \tau$. Here $\tau > 0$ still has to be chosen. We may choose it as small as we need, but the choice has to be independent of ϵ.

Let us assume that the elements $x_1, ..., x_{j-1}$ lie in $B_\rho(x_0)$. As F is bounded on $[t_0, t_0+\rho] \times B_\rho(x_0)$, we obtain from (A.1), (A.2), (A.3) that

$$\|x_j - T_0(t_j - t_{j-1})x_{j-1}\| \leq M(t_j - t_{j-1}) \tag{A.4}$$

with some constant $M > 0$. Using this inequlity twice we obtain

$$\begin{aligned} &\|x_j - T_0(t_j - t_{j-2})x_{j-2}\| \\ &\leq \|x_j - T_0(t_j - t_{j-1})x_{j-1}\| + \|T_0(t_j - t_{j-1})\big(x_{j-1} - T_0(t_{j-1} - t_{j-2})x_{j-2}\big)\| \\ &\leq M(t_j - t_{j-1}) + Me^{\omega(t_j - t_{j-1})}(t_{j-1} - t_{j-2}) \\ &\leq Me^{\omega(t_j - t_{j-1})}(t_j - t_{j-2}). \end{aligned}$$

Here we have assumed $\omega > 0$ without restriction of generality. Continuing like this we obtain

$$\|x_j - T_0(t_j - t_k)x_k\| \leq Me^{\omega(t_j - t_k)}(t_j - t_k) \tag{A.5}$$

for $k < j$. So choosing the number $\tau \in (0, \rho)$ small enough we find that

$$x_j \in B_\rho(x_0) \quad \text{if} \quad t_j - t_0 \leq \tau. \tag{A.6}$$

Note that the choice of τ is independent of ϵ as long as $\epsilon \leq 1$.

So, by induction, we can conclude that (A.1),...,(A.6) hold for all j as long as $t_j \leq t_0 + \tau$.

Step 2: We claim that, after finitely many steps, $t_j = t_0 + \tau$ for some j.

If this is not the case, we find a convergent sequence

$$t_j \to t \leq t_0 + \tau, \; j \to \infty, \; t_j < t$$

and a sequence of elements $x_j \in C_0$ such that (A.1),...,(A.6) hold for all j. We claim that the elements x_j form a Cauchy sequence and thus converge towards some $x \in C_0$. In order to realize this let k be arbitrary and $j, i > k$. Then

$$\|x_j - x_i\|$$

$$\leq \|x_j - T_0(t_j - t_k)x_k\| + \|T_0(t_j - t_k)x_k - T_0(t_i - t_k)x_k\| + \|x_i - T_0(t_i - t_k)x_k\|$$

$$\leq Me^{\omega\tau}[t_j - t_k + t_i - t_k] + \|T_0(t_j - t_k)x_k - T_0(t_i - t_k)x_k\|.$$

Here we have used (A.5). Keeping k fixed we obtain

$$\limsup_{i,j\to\infty} \|x_j - x_i\| \leq 2Me^{\omega\tau}[t - t_k].$$

Here we have used the strong continuity of the semigroup. As k was arbitrary, we can now take the limit for $k \to \infty$ and obtain

$$\limsup_{i,j\to\infty} \|x_j - x_i\| = 0$$

Let

$$x = \lim_{j\to\infty} x_j.$$

As the times t_j were maximally chosen and $t > t_j$ we have from (A.1), (A.2), (A.3) that

$$dist\left(T_0(t - t_j)x_j - \lim_{\lambda\to\infty} \int_{t_j}^{t} T_0(t - s)\lambda(\lambda - A)^{-1}F(t_j)x_j ds; C_0 \right) \geq \frac{\epsilon}{2}(t - t_j)$$

or that there exist $s_j \in [0, t - t_j]$ such that

$$\|T_0(s_j)x_j - x_j\| \geq \epsilon.$$

The first inequality contradicts the subtangential condition in lemma 2.9, the second the strong continuity of the semigroup T_0.

Step 3: Construction of the polygonal approximate solution

It follows from (A.2),..., (A.4) that

$$\|x_j - x_{j-1}\| \leq (M + 1)\epsilon. \tag{A.7}$$

We define

$$x(t) = \frac{t - t_{j-1}}{t_j - t_{j-1}} x_j + \frac{t_j - t}{t_j - t_{j-1}} x_{j-1}$$

for $t_{j-1} \leq t \leq t_j$. $x(t) \in C_0$ because elements of a convex set have been combined in a convex manner. Further

$$\|x(t) - x_{j-1}\| \leq \|x_j - x_{j-1}\| \leq (M + 1)\epsilon. \tag{A.8}$$

It now follows from the Lipschitz condition and the continuity of F that

$$(A.9)\quad \|x_j - T_0(t_j - t_{j-1})x_{j-1} - \lim_{\lambda\to\infty}\int_{t_{j-1}}^{t_j} T_0(t_j - s)\lambda(\lambda - A)^{-1}F(s)x(s)ds\| \le c\epsilon(t_j - t_{j-1})$$

with an appropriate constant $c > 0$. Applying this inequality twice we obtain

$$\|x_j - T_0(t_j - t_{j-2})x_{j-2} - \lim_{\lambda\to\infty}\int_{t_{j-2}}^{t_j} T_0(t_j - s)\lambda(\lambda - A)^{-1}F(s)x(s)ds\|$$

$$\le \|x_j - T_0(t_j - t_{j-1})x_{j-1} - \lim_{\lambda\to\infty}\int_{t_{j-1}}^{t_j} T_0(t_j - s)\lambda(\lambda - A)^{-1}F(s)x(s)ds\|$$

$$+\|T_0(t_j - t_{j-1})\|\ \|x_{j-1} - T_0(t_{j-1} - t_{j-2})x_{j-2} - \lim_{\lambda\to\infty}\int_{t_{j-2}}^{t_{j-1}} T_0(t_{j-1} - s)\lambda(\lambda - A)^{-1}F(s)x(s)ds\|$$

$$\le c\epsilon e^{\omega(t_j - t_{j-2})}(t_j - t_{j-2}).$$

Proceeding in this way we find

$$\|x_j - T_0(t_j - t_0)x_0 - \lim_{\lambda\to\infty}\int_{t_0}^{t_j} T_0(t_j - s)\lambda(\lambda - A)^{-1}F(s)x(s)ds\|$$

$$\le c\epsilon e^{\omega(t_j - t_0)}(t_j - t_0).$$

In a similar way, by using (A.1), (A.2), (A.3), and (A.8), we find, if $t_j \le t \le t_{j+1} \le t_0 + \tau$,

$$\|x(t) - T_0(t - t_0)x_0 - \lim_{\lambda\to\infty}\int_{t_0}^{t} T_0(t - s)\lambda(\lambda - A)^{-1}F(s)x(s)ds\|$$

$$\le \lim_{\lambda\to\infty}\|\int_{t_j}^{t} T_0(t - s)\lambda(\lambda - A)^{-1}F(s)x(s)ds\|$$

$$+\|T_0(t - t_j)\|\ \|x_j - T_0(t_j - t_0)x_0 - \lim_{\lambda\to\infty}\int_{t_0}^{t_j} T_0(t_j - s)\lambda(\lambda - A)^{-1}F(s)x(s)ds\|$$

$$+\|x(t) - x_j\| + \|T_0(t - t_j)x_j - x_j\|$$

$$\le c\epsilon$$

with a possibly larger constant c than before. τ and c are independent of ϵ.

Step 4: Approximation of the solution

The polygonal approximate solution x we have just found is called an ϵ-solution. Let x be an ϵ-solution and y be a δ solution. Then, by the last inequality and the Lipschitz condition for F,

$$\|x(t) - y(t)\| \leq \int_0^t e^{\omega(t-s)} \Lambda \|x(s) - y(s)\| ds + (\epsilon + \delta)c$$

for $t_0 \leq t \leq t_0 + \tau$ with an appropriate constant c. A Gronwall argument implies that the ϵ-solutions form a Cauchy-system in the supremum norm for $\epsilon \to 0$. So they have a uniform limit u which is a solution of ($\spadesuit$) on $[t_0, t_0 + \tau]$.

The uniqueness of u follows from the Lipschitz condition and a Gronwall argument in a similar way.

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