Analysis of Age-Structured Population Models with an Additional Structure

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Abstract

It is illustrated that age-structured population models with an additional structure nicely fit into the framework of Lipschitz perturbations of non-dense operators which satisfy the resolvent estimates. This makes it not only possible to show that unique solutions to the model equations exist in a well-defined sense, but also that they generate a dynamical system (or semiflow) which leaves closed convex subsets invariant that satisfy an appropriate subtangential condition.

1. Introduction

We consider the age-structured population model

$$
\begin{aligned}
(\partial_t + \partial_a)u(t,a) &= A(a)u(t,a) + F(a)u(t,\cdot); \quad t, a > 0. \\
u(t,0) &= G(u(t,.)); \quad t > 0. \\
u(0,a) &= u_0(a); \quad a \geq 0.
\end{aligned}
\tag{$\diamondsuit$}
$$

The solution $u(t,a)$ of ($\diamondsuit$) we look for takes values in a Banach space E. The Banach space E contains an additional structure of the population, whereas the age-structure has been made explicit in ($\diamondsuit$). How the individuals change with respect to the additional population structure (which may be the size or/and the spatial distribution of the individuals) is described by $A(a)$. Note that the rate of change may depend on the age of the individual. Individual mortality rates and/or

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immigration are described by nonlinear Lipschitz continuous operators $F(a)$ on E. The population birth rate is given by a nonlinear operator G from

$$L_1 = L_1([0,\infty), E)$$

into E. The integrable functions L_1 on $[0,\infty)$ with values in E are the right space for $u(t,a)$ considered as a function of age because the L_1-norm gives the population size.

Concrete examples for $(\diamondsuit)$ are age-dependent spatial spread on a bounded open domain $\Omega \subset \mathbf{R}^n$, $n = 2, 3$,

$$\begin{aligned}
(\partial_t + \partial_a)u(t,a,\omega) &= \sum_{j,k=1}^{n} \partial_{\omega_j}\Big(d_{jk}(a,\omega)\partial_{\omega_k}u(t,a,\omega)\Big) + F(a,\omega,u(t,\cdot)); \\
&\qquad t,a>0, \omega\in\Omega \\
u(t,0,\omega) &= G(\omega, u(t,.)); \quad t>0, \omega\in\Omega \\
u(0,a,\omega) &= u_0(a,\omega); \qquad a\geq 0, \omega\in\Omega \\
\partial_\nu u(t,a,\omega) &= 0; \qquad\qquad t,a>0, \omega\in\partial\Omega
\end{aligned}$$

or age and size dependent population growth

$$\begin{aligned}
(\partial_t + \partial_a)u(t,a,\sigma) &= \partial_\sigma\Big(g(a,\sigma)u(t,a,\sigma)\Big) + F(a,\sigma,u(t,\cdot)); \quad t,a,\sigma>0. \\
u(t,0,\sigma) &= G(\sigma, u(t,.)); \quad t,\sigma>0. \\
u(0,a,\sigma) &= u_0(a,\sigma); \qquad a\geq 0, \sigma>0.
\end{aligned}$$

or combinations of these.

Though age-dependent diffusion has been considered by *Di Blasio* (1979), *Gurtin* (1973), *Gurtin&McCamy* (1979), *Kunisch et al.* (1985), *Langlais* (1985, 1988), *Marcati* (1981), *Marcati&Serafini* (1979), *Webb* (1982), and others (see these references and *Webb* (1985) for more information) the general case that the diffusion operator depends both on age and space does not seem to have attracted much interest. Age and size dependent population growth has been dealt with by *Tucker&Zimmerman* (1989).

It is the aim of this paper to show how such models fit into the framework of Lipschitz perturbations of non-dense operators which satisfy the resolvent estimates. This theory which has started to be developed in *Thieme* (to appear, b) allows the unified analysis of these and many other models in population dynamics.

Here we restrict ourselves to showing the unique existence of solutions to ($\diamondsuit$) which includes the nontrivial task of explaining in which precise sense ($\diamondsuit$) can be satisfied. Since only certain subsets of L_1 — the non-negative functions, e.g. — may make sense for the model, we also look for conditions under which these subsets are invariant under the solution flow. It readily follows from the theory developed in *Thieme* (to appear, b) that

$$(T(t)u_0)(a) = u(t, a)$$

provides a semiflow (dynamical system, nonlinear semigroup) on L_1. *Thieme* (to appear, b) also provides conditions for the regularity of solutions and the stability or instability of equilibria which can be translated for ($\diamondsuit$) but are not worked out here. Further development of the theory will hopefully provide applicable conditions for the existence of periodic solutions, for asymptotically stable exponential growth in linear population models (see *Marcati&Serafini*, 1979), for existence of open dense sets of convergent points (see *Hirsch* (1988), *Matano* (1987), *Smith&Thieme* (to appear, submitted)), etc.

An alternative approach consists in considering ($\diamondsuit$) as an abstract semilinear boundary value problem (see *Greiner*, (1987, 1989). But the theory of non-dense operators which bases on the work by *da Prato&Sinestrari* (1987) and the theory of 'integrated semigroups' (*Arendt* (1987a,b), *Kellermann* (1986), *Kellermann&Hieber* (1989), *Neubrander* (1989), *Thieme* (to appear, a)) seems to lead more directly to the precise meaning of the operator $-\frac{d}{da} + A(\cdot)$.

2. Motivation of the approach

Let us consider the nonlinear operator

$$\mathcal{H}_0 x(a) = -\frac{d}{da}x(a) + A(a)x(a) + F(a)x$$

defined on a suitable subset $D(\mathcal{H}_0)$ of L_1 which we do not prescribe precisely here except that it contains the boundary condition $x(0) = G(x)$, i.e.

$$D(\mathcal{H}_0) \subseteq \{x \in L_1; x \text{ continuous}, \ x(0) = G(x)\}.$$

Our method consists in writing both $F(a)$ and G as bounded perturbations of an extension of the operator $-\frac{d}{da} + A(\cdot)$.

To this end we consider the Banach space

$$X = E \times L_1$$

and the closed subspace

$$X_0 = \{0\} \times L_1.$$

We set

$$\mathcal{A}(0, x) = (-x(0), -\frac{d}{da}x + A(\cdot)x(\cdot)),$$

without explaining the precise meaning and domain of definition now, and

$$\mathcal{F}x = (Gx, F(\cdot)x).$$

The intention of this construction becomes clear by noticing that $\mathcal{H}_0$ is the part of the operator

$$\mathcal{H} = \mathcal{A} + \mathcal{F}$$

in X_0.

Apparently the operator $\mathcal{A}$ is not densely defined, but it will turn out that $\mathcal{A}$ satisfies the resolvent estimates of semigroup theory. Our procedure consists in applying the theory of Lipschitz perturbations of non-densely defined operators which satisfy the resolvent estimates (*Thieme*, to appear, b).

3. A semigroup on X_0 and an 'integrated semigroup' on X

First of all the definition of the operator $-\frac{d}{da} + A(\cdot)$ in L_1 — see ($\diamondsuit$) — and of the operator $\mathcal{A}$ in X have to be made precise.

We assume that there exists an evolutionary system $U(t, s)$ with the following properties:

Assumption 3.1.

(i) *The family* $U(t, s), t \geq s \geq 0$, *consists of bounded linear operators.*

(ii) $U(t, s)U(s, r) = U(t, r)$ *if* $t \geq s \geq r \geq 0$.

(iii) $U(s, s)x = x$ *if* $x \in E, s \geq 0$.

(iv) $U(t, s)x$ *is a continuous mapping of* t *for* $t \geq s \geq 0$ *and* $x \in E$.

(v) *There is a constant* $c > 0$ *such that* $\|U(a + s, a)\| \leq c,\ 0 \leq s \leq 1, a \geq 0$.

(vi) $U(t, s)x$ *is a Bochner measurable function of* $s, 0 \leq s \leq t$.

We tentatively define the operator $-\frac{d}{da}+A(\cdot)$ in $(\diamondsuit)$ and the operator $\mathcal{A}$:

$$(-\frac{d}{da}+A(a))x(a)=\lim_{h\downarrow 0}\frac{1}{h}\Big(U(a,a-h)x(a-h)-x(a)\Big) \text{ for a.a } a>0 \tag{3.1}$$

provided that this limit makes sense and is integrable.

$$D(\mathcal{A})=\{(0,x);x\in D(-\frac{d}{da}+A(\cdot))\}$$

$$\mathcal{A}(0,x)=(-x(0),(-\frac{d}{da}+A(\cdot))x). \tag{3.2}$$

The definition of $\mathcal{A}$ takes for granted that any element $x\in D(-\frac{d}{da}+A(\cdot))$ is continuous at $a=0$.

The definitions of $-\frac{d}{da}+A(\cdot)$ and $\mathcal{A}$ are motivated by the fact that

$$(T_0(t)x)(a)=\begin{cases}U(a,a-t)x(a-t) & \text{if } a\geq t\\ 0 & \text{if } a<t\end{cases} \tag{3.3}$$

defines a strongly continuous semigroup on L_1 and are modeled in analogy to the generator $\mathcal{A}_0$ of T_0. Actually $\mathcal{A}_0$ will turn out to be the restriction of $-\frac{d}{da}+A(\cdot)$ to those elements in $D(-\frac{d}{da}+A(\cdot))$ with $x(0)=0$. It is convenient to identify T_0 with a strongly continuous semigroup on X_0 by setting

$$T_0(t)(0,x)=(0,T_0(t)x).$$

Then $\mathcal{A}_0$ can be identified with the part of $\mathcal{A}$ in X_0.

Let us formally determine $(\lambda-\mathcal{A})^{-1}$. Let

$$(\lambda-\mathcal{A})(0,u)=(\eta,\tilde{u})$$

i.e. u be a solution of

$$\lambda u(a)+u'(a)-A(a)u(a)=\tilde{u}(a),u(0)=\eta.$$

Analogy with non-autonomous differential equations suggests to try

$$u(a)=e^{-\lambda a}U(a,0)\eta+\int_0^a e^{\lambda s}U(a,a-s)\tilde{u}(a-s)ds. \tag{3.4}$$

Recalling the relation between the resolvent of the generator and the Laplace integral of a semigroup, we realize that the integral on the right hand side of this equation is the resolvent of T_0. As

$$e^{-\lambda a}U(a,0)\eta=\lambda\int_a^\infty e^{-\lambda t}U(a,0)\eta dt,$$

we have, formally,

$$(\lambda - \mathcal{A})^{-1}(\eta, \tilde{u}) = \lambda \int_0^\infty e^{-\lambda t} S(t)(\eta, \tilde{u}) dt \tag{3.5}$$

with

$$S(t)(\eta, \tilde{u}) = \Big(0, a \mapsto H(t-a)U(a,0)\eta + \int_0^t (T_0(s)\tilde{u})(a)ds\Big). \tag{3.6}$$

Here H denotes the Heaviside function. We now start to make our formal considerations rigorous.

Lemma 3.2. *The family S of operators defined in (3.6) is an 'integrated semigroup' which is locally Lipschitz continuous in the uniform operator topology.*

An 'integrated semigroup' is a family of bounded linear operators $S(t), t \geq 0$, satisfying the relations

$$S(t)S(r) = \int_0^t (S(r+\tau) - S(\tau))d\tau, \quad S(0) = 0.$$

This definition is motivated by the fact that any strongly continuous semigroup T_0 gives rise to an integrated semigroup S_0 via $S_0(t) = \int_0^t T_0(s)ds$.

Proof: It is sufficient to check the functional equation for $x = 0$ because, for $\xi = 0$, S is nothing else than the integrated semigroup originating from integrating T_0. Then the second component of $S(r)S(t)(\xi, 0)$ is

$$\int_0^{\min(r,a)} U(a, a-s)S(t)(\xi,0)(a-s)ds$$

$$= \int_0^{\min(r,a)} U(a, a-s)H(t+s-a)U(a-s,0)\xi ds$$

$$= \int_0^{\min(r,a)} H(t+s-a)dsU(a,0)\xi$$

$$= \int_0^r [H(t+s-a) - H(s-a)]dsU(a,0)\xi$$

But this is the second component of $\int_0^r [S(t+s) - S(s)](\xi, 0)ds$.

The local Lipschitz continuity needs checking only for $x = 0$ because, for $\xi = 0$, S is just the integrated semigroups originating from T_0. Now

$$\int_0^\infty \|\Big(H(t-a) - H(s-a)\Big)U(a,0)\xi\|da \leq \int_s^t \|U(a,0)\| \, \|\xi\|da$$

$$\leq \sup_{s\leq a\leq t} \|U(a,0)\| \, \|\xi\|(t-s).$$

For ‘integrated semigroups’ a *generator* $\mathcal{A}$ can be defined in analogy to semigroups:

$$u \in D(\mathcal{A}),\ \mathcal{A}u = v \iff \frac{d}{dt}S(t)u = u + S(t)v,\ t \geq 0.$$

It has been shown by *Thieme* (to appear, a) that relation (3.5) holds between the resolvent of the generator $\mathcal{A}$ and the Laplace transform of an ‘integrated semigroup’ S. Note the analogy with semigroups. Hence we can rigorously derive for the generator $\mathcal{A}$ of S that

$$(\lambda - \mathcal{A})^{-1}(\eta, \tilde{u}) = \left(0, a \mapsto e^{-\lambda a}U(a,0)\eta + \int_0^a e^{\lambda s}U(a,a-s)\tilde{u}(a-s)ds\right). \qquad (3.8)$$

We notice that this expression makes formal sense for $\lambda = 0$ and so we guess that the generator $\mathcal{A}$ of S can be characterized as follows:

$$(\xi, x) \in D(\mathcal{A}), \quad \mathcal{A}(\xi, x) = (\eta, y)$$

iff

$$\xi = 0, \quad x(a) = -U(a,0)\eta - \int_0^a U(a,s)y(s)ds. \qquad (3.9)$$

It follows that any x with $(0,x) \in D(\mathcal{A})$ is continuous and $x(0) = -\eta$. Before we prove (3.9) we notice the following.

Lemma 3.3. *$S(t)(\xi, x)$ is continuously differentiable in t iff $\xi = 0$, and $S'(t)(0,x) = (0, T_0(t)x)$.*

With this lemma in mind we translate the definition of the generator of an ‘integrated semigroup’ to S. First we notice that $D(\mathcal{A}) \subset X_0$. Further

$$(0,x) \in D(\mathcal{A}), \mathcal{A}(0,x) = (\eta, y)$$

iff

$$U(a,a-t)x(a-t)$$

$$= x(a) + H(t-a)U(a,0)\eta + \int_0^{\min(t,a)} U(a,a-s)y(a-s)ds \qquad (3.10)$$

Choosing $t > a$ we obtain (3.9). As this is possible for any a, (3.9) is a necessary condition for (3.10) to hold. It is also sufficient for (3.10) to hold, if $t > a$. Let now $t \le a$. Then, by using (3.9) twice,

$$U(a, a-t)x(a-t) = U(a, a-t)\Big(-U(a-t, 0)\eta - \int_0^{a-t} U(a-t, s)y(s)ds\Big)$$

$$= -U(a, 0)\eta - \int_0^{a-t} U(a, s)y(s)ds = x(a) + \int_{a-t}^{a} U(a, s)y(s)ds$$

$$= x(a) + \int_0^t U(a, a-s)y(a-s)ds)$$

But this is (3.10) for $t \le a$.

Proposition 3.4.

$$(\xi, x) \in D(\mathcal{A})$$

if and only if $\xi = 0$, x *is continuous, and*

$$x(a) = U(a, 0)x(0) - \int_0^a U(a, s)y(s)ds$$

for some $y \in L_1$.

Moreover

$$\mathcal{A}(0, x) = (-x(0), y).$$

Actually it is possible to describe $\mathcal{A}$ explicitly.

Corollary 3.5. *a) Let* $x \in L_1$ *and* $(0, x) \in D(\mathcal{A})$. *Then* x *is continuous and* $\mathcal{A}(0, x) = (-x(0), y)$ *with*

$$y(a) = \lim_{h \downarrow 0} \frac{1}{h}\Big(U(a, a-h)x(a-h) - x(a)\Big)\Big) \quad \text{for a.a. } a > 0.$$

b) Let $x, y \in L_1$, x *continuous and*

$$y(a) = \lim_{h \downarrow 0} \frac{1}{h}\Big(U(a, a-h)x(a-h) - x(a)\Big)\Big) \quad \text{for a.a. } a > 0.$$

Then $(0, x) \in D(\mathcal{A})$ *and* $\mathcal{A}(0, x) = (-x(0), y)$.

Proof: Let $\mathcal{A}(0,x) = (-x(0), y)$. By proposition 3.4,

$$x(a)$$

$$= U(a,a-h)\left(U(a-h,0)x(0) - \int_0^{a-h} U(a-h,s)y(s)ds\right) - \int_{a-h}^{a} U(a,s)y(s)ds$$

$$= U(a,a-h)x(a-h) - \int_{a-h}^{a} U(a,s)y(s)ds.$$

Hence

$$\|\frac{1}{h}\Big(U(a,a-h)x(a-h) - x(a)\Big) - y(a)\| \le \frac{1}{h}\int_{a-h}^{a} \|U(a,s)y(s) - y(a)\|ds$$

$$\le \frac{1}{h}\int_{a-h}^{a} \|U(a,s)\|\, \|y(s) - y(a)\|ds + \frac{1}{h}\int_{a-h}^{a} \|U(a,s)y(a) - y(a)\|ds$$

$$\le c\frac{1}{h}\int_{a-h}^{a} \|y(s) - y(a)\|ds + \frac{1}{h}\int_{a-h}^{a} \|U(a,s)y(a) - y(a)\|ds.$$

with a suitable constant c which we obtain from the uniform boundedness theorem. Convergence to 0 a.e. now follows from theorem 3.8.5 and its corollary 2 in *Hille&Phillips* (1957) and from the strong continuity of U.

Let now y be the limit in the statement of the corollary. Then

$$\int_0^a U(a,s)y(s)ds = \lim_{h\downarrow 0}\frac{1}{h}\int_h^a \Big(U(a,s-h)x(s-h) - U(a,s)x(s)\Big)ds$$

$$= \lim_{h\downarrow 0}\frac{1}{h}\left(\int_0^{a-h} U(a,s)x(s)ds - \int_h^a U(a,s)x(s)ds\right)$$

$$= \lim_{h\downarrow 0}\frac{1}{h}\left(-\int_{a-h}^{a} U(a,s)x(s)ds + \int_0^h U(a,s)x(s)ds\right)$$

$$= -x(a) + U(a,0)x(0)$$

for a.a. $a > 0$. The last conclusion holds though we have not assumed that $U(a,s)$ is strongly continuous in s. To make it precise one has actually to take the L_1-norm of the left and the right hand side of this equation and to use the strong continuity of $U(a,s)$ in a.

Now we can give a precise meaning to the operator $-\frac{d}{da} + A(\cdot)$ — see (3.2):

Definition 3.6. *Let $x, y \in L_1$. Then $x \in D(-\frac{d}{da} + A(\cdot))$ and $y(a) = (-\frac{d}{da} + A(\cdot))x(a)$ iff x is continuous and*

$$x(a) = U(a,0)x(0) - \int_0^a U(a,s)y(s)ds.$$

The justification of our tentative definition (3.1) follows from corollary 3.5.

Corollary 3.7. *Let $x, y \in L_1$. Then $x \in D(-\frac{d}{da} + A(\cdot))$ and $y(a) = (-\frac{d}{da} + A(\cdot))x(a)$ iff x is continuous and*

$$y(a) = \lim_{h \downarrow 0} \frac{1}{h}\Big(U(a, a-h)x(a-h) - x(a)\Big) \text{ for a.a. } a > 0.$$

4. Integrated solutions of inhomogeneous linear problems and the variation of constants formula

With the procedure of section 3 the inhomogeneous linear equation

$$\begin{aligned} (\partial_t + \partial_a)u(t,a) &= A(a)u(t,a) + f(t,a); \quad t, a > 0. \\ u(t,0) &= g(t); \quad t > 0. \\ u(0,a) &= u_0(a); \quad a \geq 0. \end{aligned} \tag{4.1}$$

can be rigorously formulated as

$$v(t) = v_0 + \mathcal{A}\int_0^t v(s)ds + \int_0^t h(s)ds. \tag{$\heartsuit$}$$

with

$$v(t) = (0, u(t,\cdot),\ v_0 = (0, u_0),\ h(t) = (g(t), f(t,\cdot).$$

($\heartsuit$) is uniquely solved by the variation of constants formula

$$\begin{aligned} v(t) &= S'(t)v_0 + \frac{d}{dt}\int_0^t S(t-s)h(s)ds \\ &= T_0(t)x_0 + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)\lambda(\lambda - \mathcal{A})^{-1}h(s)ds. \end{aligned} \tag{$\spadesuit$}$$

See *Kellermann&Hieber* (1989) and *Thieme* (to appear, b). By using our characterization of $\mathcal{A}$ in proposition 3.4 (♡) becomes equivalent to

$$\int_0^t u(s,a)ds$$

$$= U(a,0)\int_0^t g(s)ds + \int_0^a U(a,a')\Big(-u(t,a') + u_0(a') + \int_0^t f(s,a')ds\Big)da',$$

i.e. to

$$\begin{aligned}\int_0^t u(s,0)ds &= \int_0^t g(s)ds \\ u(t,a) &= U(a,0)g(t) - \frac{d}{dt}\int_0^a U(a,a')u(t,a')da' + \int_0^a U(a,a')f(t,a')da' \\ u(0,a) &= u_0(a)\end{aligned} \qquad (\heartsuit\heartsuit)$$

for a.a. a.

Another way of giving (4.1) a precise meaning is the following:

$$\begin{aligned}\int_0^t u(s,0)ds &= \int_0^t g(s)ds \\ u(t,a) - u_0(a) &= (-\frac{d}{da} + A(\cdot))\int_0^t u(s,a)ds + \int_0^t f(s)ds\end{aligned} \qquad (4.2)$$

with the operator $-\frac{d}{da} + A(\cdot)$ being defined by definition 3.6 and equivalently described by corollary 3.7.

Translating (♠) yields

$$u(t,a) = \begin{cases} U(a,a-t)x_0(a-t) + \int_0^t U(a,a-s)f(t-s,a-s)ds & \text{if } a > t \\ \int_0^a U(a,a-s)f(t-s,a-s)ds + U(a,0)g(t-a) & \text{if } a < t \end{cases} \qquad (♠♠)$$

for a.a. a. This corresponds to the formula one obtains by integrating (4.1) along characteristic lines.

5. Solutions to the semilinear problem and invariance of closed convex sets

Following section 4 we can rewrite (♢) as

$$v(t) = v_0 + \mathcal{A}\int_0^t v(s)ds + \int_0^t \mathcal{F}v(s)ds. \qquad (5.1)$$

with

$$v(t) = (0, u(t,\cdot)),\ v_0 = (0, u_0),\ \mathcal{F}(0,u) = (G(u), F(\cdot)u). \tag{5.2}$$

Formulations of (5.1) which are closer to ($\diamondsuit$) can be translated from (4.2):

$$\begin{aligned} \int_0^t u(s,a)ds &= \int_0^t G(u(s))ds \\ u(t,a) - u_0(a) &= \left(-\frac{d}{da} + A(\cdot)\right)\int_0^t u(s,a)ds \ + \ \int_0^t F(a)u(s,\cdot)ds \end{aligned} \tag{5.3}$$

with the operator $-\frac{d}{da} + A(\cdot)$ being defined by definition 3.6 and equivalently described by corollary 3.7, or from ($\heartsuit\heartsuit$). Solutions to (5.1) are equivalent to solutions to the equation

$$v(t) = T_0(t)v_0 + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)\lambda(\lambda - \mathcal{A})^{-1}\mathcal{F}v(s)ds. \tag{5.4}$$

(5.4) is useful in many respects, e.g. for proving the existence and uniqueness of solutions to (5.1), in particular if the equations ($\diamondsuit$) only make sense in a subset $\mathcal{C}_0$ of L_1 such that one has to guarantee that the solutions stay in $\mathcal{C}_0$.

In the following we assume that $\mathcal{C}_0$ is a closed convex subset of L_1.

We make the following

Assumptions 5.1.

a) Let the evolutionary system $U(t,s)$ associated with the operator $\mathcal{A}$ satisfy the assumptions 3.1.

b) Let the operators $F(a) : \mathcal{C}_0 \to E$ satisfy

$$\int_0^\infty \|F(a)u\|da \le c\left(1 + \int_0^\infty \|u(a)\|da\right) \text{ for } u \in \mathcal{C}_0$$

and

$$\int_0^\infty \|F(a)u - F(a)\tilde{u}\|da \le c\int_0^\infty \|u(a) - \tilde{u}(a)\|da \quad \text{for } u, \tilde{u} \in \mathcal{C}_0$$

with some constant $c > 0$.

c) Let the operator $G : \mathcal{C}_0 \to E$ satisfy

$$\|G(u) - G(\tilde{u})\| \le c\|u - \tilde{u}\| \text{ for } u, \tilde{u} \in \mathcal{C}_0.$$

The subsequent result is a consequence of theorem 2.4 by *Thieme* (to appear, b).

Proposition 5.2. *Under the assumptions 5.1 there exists a unique solution to (5.3) with values in $\mathcal{C}_0$ for any $u_0 \in \mathcal{C}_0$ iff the following subtangential condition is satisfied for any $u_0 \in \mathcal{C}_0$:*

$$\frac{1}{h}\text{dist}\Big(T_0(t)u_0 + S(h)\mathcal{F}(0,u_0);\mathcal{C}_0\Big) \to 0, \quad h \downarrow 0. \tag{5.5}$$

Here T_0 and S are the semigroup and 'integrated semigroup' associated with the evolutionary system U via (3.3) and (3.6). (5.5) then takes the form

$$\frac{1}{h}\text{dist}\left(T_0(t)u_0 + \int_0^h T_0(s)F(\cdot)u_0 ds + H(t-\cdot)U(\cdot,0)G(u_0);\mathcal{C}_0\right) \to 0, \quad h \downarrow 0.$$

Actually it seems to be more useful to replace $\int_0^h T_0(s)F(\cdot)u_0 ds$ by $hT_0(h)F(\cdot)u_0$ in this formula. Using (3.3) we obtain the following form of the subtangential condition.

Theorem 5.3. *Under the assumptions 5.1 there exists a unique solution to (5.4) with values in $\mathcal{C}_0$ for any $u_0 \in \mathcal{C}_0$ iff the following subtangential condition is satisfied for any* $u_0 \in \mathcal{C}_0$:

$$\frac{1}{h}\text{dist}\Big(a \mapsto H(a-h)U(a,a-h)\Big(u_0(a) + hF(a)u_0\Big) + H(h-a)U(a,0)G(u_0);\mathcal{C}_0\Big) \longrightarrow 0$$

for $h \downarrow 0$.

The following special case seems to be of particular interest: Let C_0 be a closed convex set in E, $0 \in C_0$. We consider the closed convex subset $\mathcal{C}_0$ in L_1 defined by

$$\mathcal{C}_0 = \{u \in L_1; u(a) \in C_0 \text{ for a.a. } a > 0\}.$$

Corollary 5.4. *Let the assumptions 5.1 be satisfied, $u_0 \in \mathcal{C}_0$. Then there exists a unique solution to (5.4) with values in $\mathcal{C}_0$ if the following holds:*

a) $U(t,s)C_0 \subseteq C_0$, $t \geq s \geq 0$.

b) F *satisfies the subtangential condition*

$$\frac{1}{h}\text{dist}\left(u + hF(\cdot)u, \mathcal{C}_0\right) \to 0, \quad h \downarrow 0, u \in \mathcal{C}_0.$$

c) $G(C_0) \subseteq C_0$.

It follows from section 3 in *Thieme* (to appear, b) that the definition $T(t)u_0 = u(t)$ defines a dynamical system (strongly continuous nonlinear semigroup) on L_1. Section 3 also provides regularity results. Section 4 in *Thieme* (to appear, b) provides conditions for the stability and instability of equilibria which can be translated to this model.

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