Chapter 2 Solutions (Total = 34 points)

§2.1 # 2.1.6 (2 pts), 2.1.13 (2 pts), 2.1.17 (2pts)

2.1.6: Define $z \in \ell^2(\mathbb{Z}_{512})$ by

$$z(n) = 3\sin(2\pi \cdot 7n/512) - 4\cos(2\pi \cdot 8n/512).$$

Find $\hat{z}$.

Solution: The expansion $z(n) = \frac{1}{512}\sum_{m=0}^{511} \hat{z}(m)e^{2\pi imn/512}$ (the Fourier inversion formula in equation (2.10) with $N = 512$) is the expansion of z with respect to a basis (the Fourier basis), and hence the coefficients are unique. On the other hand, by Euler's formulae (equation (1.13)),

$$z(n) = 3\sin(2\pi 7n/512) - 4\cos(2\pi 8n/512) = 3\frac{e^{2\pi i7n/512} - e^{-2\pi i7n/512}}{2i} - 4\frac{e^{2\pi i8n/512} + e^{-2\pi i8n/512}}{2}$$

$$\frac{-3i}{2}e^{2\pi i7n/512} + \frac{3i}{2}e^{2\pi i(-7+512)n/512} - 2e^{2\pi i8n/512} - 2e^{-2\pi i(-8+512)n/512}$$

$$= \frac{1}{512}\left(-3 \cdot 256ie^{2\pi i7n/512} - 1024e^{2\pi i8n/512} - 1024e^{2\pi i504n/512} + 3 \cdot 256ie^{2\pi i505n/512}\right).$$

Hence $\hat{z}(7) = -768i, \hat{z}(8) = -1024, \hat{z}(504) = -1024, \hat{z}(505) = 768i$, and $\hat{z}(m) = 0$ for all other m between 0 and 511.

2.1.13: Suppose $z \in \ell^2(\mathbb{Z}_N)$. Define $\tilde{z} \in \ell^2(\mathbb{Z}_N)$ by $\tilde{z}(n) = \overline{z(N-n)}$, for $n = 0, 1, ..., N-1$. Prove that

$$(\tilde{z})\hat{}(n) = \overline{\hat{z}(n)},$$

for all n.

Solution: By definition,

$$(\tilde{z})\hat{}(n) = \sum_{m=0}^{N-1} \tilde{z}(m)e^{-2\pi imn/N} = \sum_{m=0}^{N-1} \overline{z(N-m)}e^{-2\pi imn/N}.$$

Change summation index by letting $k = N - m$. Then as m runs from 0 up to $N-1$, k runs from N down to 1. Therefore

$$(\tilde{z})\hat{}(n) = \sum_{k=1}^{N} \overline{z(k)}e^{-2\pi i(N-k)n/N} = \sum_{k=1}^{N} \overline{z(k)}e^{-2\pi i(-k)n/N}$$

$$= \overline{\sum_{k=1}^{N} z(k)e^{-2\pi ikn/N}} = \overline{\sum_{k=0}^{N-1} z(k)e^{-2\pi ikn/N}} = \overline{\hat{z}(n)},$$

since $e^{-2\pi i(N-k)n/N} = e^{-2\pi i Nn/N}e^{-2\pi i(-k)n/N} = 1\cdot e^{-2\pi i(-k)n/N}$, and we have used Exercise 2.1.8 at the next to last step to shift the sum to $0 \le k \le N-1$.

<u># 2.1.17</u>: i. (Two Dimensional Fourier Basis) For $m_1 \in \mathbb{Z}_{N_1}$ and $m_2 \in \mathbb{Z}_{N_2}$, define $E_{m_1,m_2} \in \ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$ (see Exercise 2.1.15 for the definition) by

$$E_{m_1,m_2}(n_1,n_2) = \frac{1}{\sqrt{N_1N_2}}e^{2\pi i m_1n_1/N_1}e^{2\pi i m_2n_2/N_2}.$$

Prove that

$$\{E_{m_1,m_2}\}_{0\le m_1\le N_1-1,\,0\le m_2\le N_2-1}$$

is an orthonormal basis for $\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$. As in the one-dimensional case, we usually renormalize by setting

$$F_{m_1,m_2}(n_1,n_2) = \frac{1}{N_1N_2}e^{2\pi i m_1n_1/N_1}e^{2\pi i m_2n_2/N_2}.$$

We call $F = \{F_{m_1,m_2}\}_{0\le m_1\le N_1-1,\,0\le m_2\le N_2-1}$ the Fourier basis for $\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$.

<u>Solution:</u> For $0 \le m \le N_1-1$, define $E^{(1)}_m \in \ell^2(\mathbb{Z}_{N_1})$ by $E^{(1)}_m(n) = N_1^{-1/2}e^{2\pi i mn/N_1}$ for $0\le n\le N_1-1$. Then by Lemma 2.2, $\{E^{(1)}_m\}_{0\le m\le N_1-1}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_{N_1})$. Similarly, for $0\le m\le N_1-1$ define $E^{(2)}_m \in \ell^2(\mathbb{Z}_{N_2})$ by $E^{(2)}_m(n) = N_2^{-1/2}e^{2\pi i mn/N_2}$ for $0\le n\le N_2-1$. Then by Lemma 2.2, $\{E^{(2)}_m\}_{0\le m\le N_2-1}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_{N_2})$. Note that

$$E^{(1)}_{m_1}(n_1)E^{(2)}_{m_2}(n_2) = \frac{1}{\sqrt{N_1}}e^{2\pi i m_1n_1/N_1}\frac{1}{\sqrt{N_2}}e^{2\pi i m_2n_2/N_2}$$

$$= \frac{1}{\sqrt{N_1N_2}}e^{2\pi i m_1n_1/N_1}e^{2\pi i m_2n_2/N_2} = E_{m_1,m_2}(n_1,n_2).$$

It follows by Exercise 2.1.16 that $\{E_{m_1,m_2}\}_{0\le m_1\le N_1-1,\,0\le m_2\le N_2-1}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$.

ii. (Two-Dimensional DFT) For $z\in\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$, define $\hat{z}\in\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$ by

$$\hat{z}(m_1,m_2) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} z(n_1,n_2)e^{-2\pi i m_1n_1/N_1}e^{-2\pi i m_2n_2/N_2}.$$

Also, for $w\in\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$, define $w^\vee\in\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$ by

$$w^\vee(n_1,n_2) = \frac{1}{N_1N_2}\sum_{m_1=0}^{N_1-1}\sum_{m_2=0}^{N_2-1} w(m_1,m_2)e^{2\pi i m_1n_1/N_1}e^{2\pi i m_2n_2/N_2}.$$

Prove that

$$z = (\hat{z})^\vee,$$

for all $z\in\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$. Deduce that $\hat{}:\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})\to\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$ is 1-1, hence invertible, with inverse $^\vee$. Hint: Use Exercise 2.1.16, and follow the reasoning in the text for the one-dimensional case.

<u>Solution:</u> Observe that for $z\in\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$,

$$\langle z, E_{m_1,m_2}\rangle = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} z(n_1,n_2)\overline{\frac{1}{\sqrt{N_1N_2}}e^{2\pi i m_1n_1/N_1}e^{2\pi i m_2n_2/N_2}}$$

$$= \frac{1}{\sqrt{N_1N_2}}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} z(n_1,n_2)e^{-2\pi i m_1n_1/N_1}e^{-2\pi i m_2n_2/N_2} = \frac{1}{\sqrt{N_1N_2}}\hat{z}(m_1,m_2),$$

or

$$\hat{z}(m_1,m_2) = \langle z, E_{m1,m_2}\rangle\sqrt{N_1N_2}.$$

Since $\{E_{m_1,m_2}\}_{0\leq m_1\leq N_1-1,\,0\leq m_2\leq N_2-1}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$, we have $z=\sum_{m_1=0}^{N_1-1}\sum_{m_2=0}^{N_2-1}\langle z, E_{m_1,m}$
or

$$z(n_1,n_2)=\sum_{m_1=0}^{N_1-1}\sum_{m_2=0}^{N_2-1}\langle z,E_{m_1,m_2}\rangle E_{m_1,m_2}(n_1,n_2)$$

$$=\sum_{m_1=0}^{N_1-1}\sum_{m_2=0}^{N_2-1}\frac{1}{\sqrt{N_1N_2}}\hat{z}(m_1,m_2)\frac{1}{\sqrt{N_1N_2}}e^{2\pi i m_1n_1/N_1}e^{2\pi i m_2n_2/N_2}=(\hat{z})^\vee(n_1,n_2),$$

for any $(n_1,n_2)\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$. Hence $z=(\hat{z})^\vee$ for any $z\in\ell^2(\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2})$.

This shows that $\hat{}$ is 1-1 (if $\hat{z}=0$ then $z=(\hat{z})^\vee=0^\vee=0$). Since $\hat{}$ is a linear transformation from a finite dimensional vector space to itself, it is also onto (by Exercise 1.4.7 (v)). Hence $\hat{}$ is invertible, and the identity $z=(\hat{z})^\vee$ shows that its inverse is $^\vee$.

§2.2 # 2.2.1 (3pts = 1+2), 2.2.3 (3pts), 2.2.4 (3 pts), 2.2.7 (2pts), 2.2.9 (2pts), 2.2.10 (2pts), 2.2.11 (iv) (2pts), 2.2.17 (2pts)

2.2.1: For $z\in\ell^2(\mathbb{Z}_N)$, define $T(z)\in(\mathbb{Z}_N)$ by

$$(T(z))\,(n)=z(n-1),$$

for all n.

i. Prove that T is translation invariant.

Solution: For $k\in\mathbb{Z}_N$,

$$(T(R_kz))(n)=(R_kz)(n-1)=z(n-1-k),$$

while

$$(R_kT(z))(n)=(T(z))(n-k)=z(n-k-1).$$

Thus $T(R_kz)=R_kT(z)$ for all $z\in\ell^2(\mathbb{Z}_N)$ and for all $k\in\mathbb{Z}$. Hence T is translation invariant.

ii. Let $w(n)=\cos(2\pi n/N)$, for $n\in\mathbb{Z}_N$. For each $N\geq 3$, show that w is not an eigenvector of T.

Remark: This shows that the orthonormal basis of sines and cosines in Exercise 2.1.7 i does not diagonalize T.

Solution: Note that

$$(T(w))(n)=w(n-1)=\cos(2\pi(n-1)/N)=\cos\left(\frac{2\pi n}{N}-\frac{2\pi}{N}\right)$$

$$=\cos(2\pi n/N)\cos(2\pi/N)+\sin(2\pi n/N)\sin(2\pi/N)=\cos(2\pi/N)w(n)+\sin(2\pi/N)\sin(2\pi n/N),$$

using the formula $\cos(\theta-\varphi)=\cos\theta\cos\varphi+\sin\theta\sin\varphi$. If w is an eigenvector of T, say $Tw=\lambda w$, we would have

$$\lambda w(n)=\cos(2\pi/N)w(n)+\sin(2\pi/N)\sin(2\pi n/N)\tag{1}$$

for all n. Letting $n=0$ in (1), and noting that $w(0)=\cos 0=1$, we get $\lambda=\cos(2\pi/N)$. Substituting this in (1) and cancelling the terms $\cos(2\pi/N)w(n)$ gives $\sin(2\pi/N)\sin(2\pi n/N)=0$ for all n. Applying this with $n=1$ yields $\sin^2(2\pi/N)=0$, which is false for $N\geq 3$. Therefore w is not an eigenvector of T.

2.2.3: Define $T:\ell^2(\mathbb{Z}_4)\to\ell^2(\mathbb{Z}_4)$ by

$$T(z)=(2z(0)-z(1),iz(1)+2z(2),z(1),0).$$

i. Let $z=(1,0,-2,i)$. Compute $T(R_1z)$ and $R_1T(z)$. Observe that they are not equal. Hence T is not translation invariant.

Solution: Since $R_1 z = (i, 1, 0, -2)$, we have

$$T(R_1 z) = (2i - 1, i \cdot 1 + 2 \cdot 0, 1, 0) = (2i - 1, i, 1, 0).$$

Also $T(z) = (2 - 0, i \cdot 0 + 2(-2), 0, 0) = (2, -4, 0, 0)$, so

$$R_1(T(z)) = (0, 2, -4, 0).$$

Obviously $T(R_1 z) \neq R_1 T(z)$.

ii. Find the matrix which represents T with respect to the standard (Euclidean) basis. Observe that it is not circulant, as we expect from i.

Solution: We have

$$T(z) = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & i & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z(0) \\ z(1) \\ z(2) \\ z(3) \end{bmatrix} = Az \quad \text{for} \quad A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & i & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the second row of A is not the translate by 1 of the first, A is not circulant.

iii. Show that $(1, i, -1, -i)$ is not an eigenvector of T. (Recall by Example 2.4 that $(1, i, -1, -i)$ is a multiple of the Fourier basis element F_1).

Solution: By definition of T,

$$T((1, i, -1, -i)) = (2 \cdot 1 - i, i \cdot i + 2(-1), i, 0) = (2 - i, -3, i, 0),$$

which is clearly not a multiple of $(1, i, -1, -i)$.

\# 2.2.4: Let $z = (2, i, 1, 0)$ and $w = (1, 0, 2i, 3)$.

i. Compute $\hat{z}$ and $\hat{w}$.

Solution: We have

$$\hat{z} = W_4 z = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 2 \\ i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 + i + 1 \\ 2 + 1 - 1 \\ 2 - i + 1 \\ 2 - 1 - 1 \end{bmatrix} = \begin{bmatrix} 3 + i \\ 2 \\ 3 - i \\ 0 \end{bmatrix}$$

and

$$\hat{w} = W_4 w = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2i \\ 3 \end{bmatrix} = \begin{bmatrix} 1 + 2i + 3 \\ 1 - 2i + 3i \\ 1 + 2i - 3 \\ 1 - 2i - 3i \end{bmatrix} = \begin{bmatrix} 4 + 2i \\ 1 + i \\ -2 + 2i \\ 1 - 5i \end{bmatrix}.$$

ii. Compute $z * w$ directly.

Solution: We get

$$z * w(0) = \sum_{n=0}^{3} z(0 - n)w(n) = z(0)w(0) + z(-1)w(1) + z(-2)w(2) + z(-3)w(3)$$

$$= z(0)w(0) + z(3)w(1) + z(2)w(2) + z(1)w(3) = 2 \cdot 1 + 0 \cdot 0 + 1 \cdot 2i + i \cdot 3 = 2 + 5i,$$

$$z * w(1) = \sum_{n=0}^{3} z(1 - n)w(n) = z(1)w(0) + z(0)w(1) + z(-1)w(2) + z(-2)w(3)$$

$$= z(1)w(0) + z(0)w(1) + z(3)w(2) + z(2)w(3) = i \cdot 1 + 2 \cdot 0 + 0 \cdot 2i + 1 \cdot 3 = 3 + i,$$

$$z*w(2)=\sum_{n=0}^{3} z(2-n)w(n)=z(2)w(0)+z(1)w(1)+z(0)w(2)+z(-1)w(3)$$
$$=z(2)w(0)+z(1)w(1)+z(0)w(2)+z(3)w(3)=1\cdot 1+i\cdot 0+2\cdot 2i+0\cdot 3=1+4i$$
$$z*w(3)=\sum_{n=0}^{3} z(3-n)w(n)=z(3)w(0)+z(2)w(1)+z(1)w(2)+z(0)w(3)$$
$$=0\cdot 1+1\cdot 0+i\cdot 2i+2\cdot 3=4.$$

So $z*w=(2+5i,3+i,1+4i,4)$.

iii. Compute $(z*w)\hat{}$ directly and check that it agrees with $\hat{z}\hat{w}$.

Solution: First,

$$(z*w)\hat{}=W_4(z*w)=\begin{bmatrix}1&1&1&1\\1&-i&-1&i\\1&-1&1&-1\\1&i&-1&-i\end{bmatrix}\begin{bmatrix}2+5i\\3+i\\1+4i\\4\end{bmatrix}$$
$$=\begin{bmatrix}2+5i+3+i+1+4i+4\\2+5i-3i+1-1-4i+4i\\2+5i-3-i+1+4i-4\\2+5i+3i-1-1-4i-4i\end{bmatrix}=\begin{bmatrix}10+10i\\2+2i\\-4+8i\\0\end{bmatrix}.$$

Second,

$\hat{z}(0)\hat{w}(0)=(3+i)(4+2i)=12+4i+6i-2=10+10i$
$\hat{z}(1)\hat{w}(1)=(2)(1+i)=2+2i$
$\hat{z}(2)\hat{w}(2)=(3-i)(-2+2i)=-6+2i+6i+2=-4+8i$
$\hat{z}(3)\hat{w}(3)=(0)(1-5i)=0,$
so that $\hat{z}\hat{w}=(10+10i,2+2i,-4+8i,0)=(z*w)\hat{}$.

2.2.7: Define $T:\ell^2(\mathbb{Z}_4)\to\ell^2(\mathbb{Z}_4)$ by

$$(T(z))(n)=3z(n-1)+z(n).$$

i. Write the matrix $A_{T,E}$ representing T with respect to the standard basis. Observe that it is circulant.

Solution: By definition,

$$(T(z))(0)=3z(-1)+z(0)=3z(3)+z(0),\qquad (T(z))(1)=3z(0)+z(1)$$
$$(T(z))(2)=3z(1)+z(2),\qquad \text{and } (T(z))(3)=3z(2)+z(3).$$

So

$$T(z)=\begin{bmatrix}1&0&0&3\\3&1&0&0\\0&3&1&0\\0&0&3&1\end{bmatrix}\begin{bmatrix}z(0)\\z(1)\\z(2)\\z(3)\end{bmatrix},\quad \text{so}\quad A_{T,E}=\begin{bmatrix}1&0&0&3\\3&1&0&0\\0&3&1&0\\0&0&3&1\end{bmatrix}.$$

ii. Find $b\in\ell^2(\mathbb{Z}_4)$ such that $T(z)=b*z$.

Solution: By Lemma 2.26, $b=[1\ 3\ 0\ 0]$, the first column in $A_{T,E}$.

iii. Find $m\in\ell^2(\mathbb{Z}_4)$ such that $T=T_{(m)}$, that is, such that $(T(z))\hat{}(n)=m(n)\hat{z}(n)$ for each n.

Solution: By Lemma 2.33, $m=\hat{b}$. So

$$m=\begin{bmatrix}1&1&1&1\\1&-i&-1&i\\1&-1&1&-1\\1&i&-1&-i\end{bmatrix}\begin{bmatrix}1\\3\\0\\0\end{bmatrix}=\begin{bmatrix}4\\1-3i\\-2\\1+3i\end{bmatrix}.$$

iv. Find the matrix $A_{T,F}$ representing T in the Fourier basis F.

Solution: By Lemma 2.34, $A_{T,F}$ is the diagonal matrix whose diagonal entries are the components of m:

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1-3i & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1+3i \end{bmatrix}.$$

v. By direct computation, check that $A_{T,E} = W_4^{-1} A_{T,F} W_4$, where W_4 is the matrix in equation (2.21).

Solution:

$$W_4^{-1} A_{T,F} W_4 = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1-3i & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1+3i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 1-3i & -3-i & -1+3i & 3+i \\ -2 & 2 & -2 & 2 \\ 1+3i & -3+i & -1-3i & 3-i \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 & 12 \\ 12 & 4 & 0 & 0 \\ 0 & 12 & 4 & 0 \\ 0 & 0 & 12 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} = A_{T,E}.$$

2.2.9: Define $T : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ by

$$(T(z))(n) = z(n+1) - z(n).$$

Find all eigenvalues of T.

Solution: We first note that T is translation invariant:

$$T(R_k z)(n) = (R_k z)(n+1) - (R_k z)(n) = z(n+1-k) - z(n-k),$$

which agrees with

$$(R_k T(z))(n) = (T(z))(n-k) = z(n-k+1) - z(n-k).$$

Therefore by Lemmas 2.26 and 2.33, the eigenvalues of T are the components of $m = \hat{b}$, where b is the first column of $A_{T,E}$. In particular, using equation (2.43),

$$b(m) = a_{m,0} = (A_{T,E} e_0)(m) = (T(e_0))(m) = e_0(m+1) - e_0(m).$$

Since $e_0 = (1,\ 0,\ 0,\ ...,\ 0)$, we get $b(m) = 0$ for $m = 1, 2, ..., N-2$. We also get

$$b(0) = e_0(1) - e_0(0) = 0 - 1 = -1 \text{ and } b(N-1) = b(N) - b(N-1) = b(0) - 0 = 1.$$

Therefore $b = (-1,\ 0,\ 0,\ ...\ 0,\ 0,\ 1)$. Hence

$$m(k) = \hat{b}(k) = \sum_{n=0}^{N-1} b(n) e^{-2\pi i k n/N} = b(0) \cdot 1 + b(N-1) e^{-2\pi i k (N-1)/N}$$

$$= -1 \cdot 1 + 1 \cdot e^{-2\pi i k(-1)/N} = -1 + e^{2\pi i k/N}.$$

Therefore the eigenvalues of T are $\{-1 + e^{2\pi i k/N}\}_{k=0}^{N-1}$.

2.2.10: Let $T_{(m)} : \ell^2(\mathbb{Z}_4) \to \ell^2(\mathbb{Z}_4)$ be the Fourier multiplier operator defined by $T_{(m)}(z) = (m\hat{z})^\vee$ where $m = (1, 0, i, -2)$.

i. Find $b \in \ell^2(\mathbb{Z}_4)$ such that $T_{(m)}$ is the convolution operator T_b (defined by $T_b(z) = b * z$).

Solution: By Lemma 2.33, $b = m^{\vee}$, or

$$b = W_4^{-1} m = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ i \\ -2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1+i \\ 1+i \\ 3+i \\ 1-3i \end{bmatrix}.$$

ii. Find the matrix that represents $T_{(m)}$ with respect to the standard basis.

Solution: By Lemma 2.26, the matrix $A_{T,E}$ that represents T with respect to the Euclidean basis must be the circulant matrix with b in its first column:

$$A_{T,E} = \frac{1}{4}\begin{bmatrix} -1+i & 1-3i & 3+i & 1+i \\ 1+i & -1+i & 1-3i & 3+i \\ 3+i & 1+i & -1+i & 1-3i \\ 1-3i & 3+i & 1+i & -1+i \end{bmatrix}.$$

2.2.11(iv): Suppose $m_1, m_2 \in \ell^2(\mathbb{Z}_N)$. Prove that the composition $T_{(m_2)} \circ T_{(m_1)}$ of the Fourier multiplier operators $T_{(m_2)}$ and $T_{(m_1)}$ is the Fourier multiplier operator $T_{(m)}$ where $m(n) = m_2(n)m_1(n)$ for all n.

Solution: For any $z \in \ell^2(\mathbb{Z}_N)$,

$$T_{(m_2)} \circ T_{(m_1)}(z) = T_{(m_2)}(T_{(m_1)}(z)) = T_{(m_2)}\left((m_1\hat{z})^{\vee}\right)$$

$$= \left(m_2\left((m_1\hat{z})^{\vee}\right)\hat{}\,\right)^{\vee} = (m_2 m_1 \hat{z})^{\vee} = (m\hat{z})^{\vee} = T_{(m)}(z),$$

using Fourier inversion to conclude that $((m_1\hat{z})^{\vee})\hat{} = m_1\hat{z}$.

2.2.17: Show that there exist $z, w \in \ell^2(\mathbb{Z}_4)$ such that $z \neq 0$ and $w \neq 0$, but $z * w = 0$, where 0 is the zero vector $(0, 0, 0, 0)$.

Solution: Let $e_0 = (1, 0, 0, 0)$ and $e_1 = (0, 1, 0, 0)$ as usual and take $z = e_0^{\vee}$ and $w = e_1^{\vee}$ (in fact by Exercise 2.3.3 below, $z = F_0$ and $w = F_1$, the first two Fourier basis vectors). Then $\hat{z} = e_0$ and $\hat{w} = e_1$. Therefore

$$(z * w)\hat{}\,(n) = \hat{z}(n)\hat{w}(n) = e_0(n)e_1(n) = 0$$

for all n, since for any n, either $e_0(n) = 0$ or $e_1(n) = 0$ (or both). Hence $(z * w)\hat{} = (0, 0, 0, 0)$. Since the DFT is one-to-one, this implies that $z * w = (0, 0, 0, 0)$.

§2.3 # 2.3.3 (2pts), 2.3.5 (2pts), 2.3.6 (2pts)

2.3.1: Trivial.

2.3.3: Let $\{e_0, e_1, ..., e_{N-1}\}$ be the Euclidean basis for $\ell^2(\mathbb{Z}_N)$, and let $\{F_0, F_1, ..., F_{N-1}\}$ be the Fourier basis.

i. Show that $\hat{e}_m(k) = e^{-2\pi imk/N}$ for all k. Notice that $\hat{e}_m$ is very nearly (up to a reflection and a normalization) an element of the Fourier basis.

Solution: By definition,

$$\hat{e}_m(k) = \sum_{n=0}^{N-1} e_m(n)e^{-2\pi ink/N} = e^{-2\pi imk/N},$$

since $e_m(n) = 0$ for $n \neq m$ whereas $e_m(m) = 1$.

ii. Show that $\hat{F}_m = e_m$.

<u>Solution:</u> By part i,

$$\hat{e}_m(k) = e^{-2\pi imk/N} = NF_m(-k).$$

Replacing k with $-k$, and recalling that $z^\vee(m) = \frac{1}{N}\hat{z}(-m)$ for any z and m, we obtain

$$F_m(k) = \frac{1}{N}\hat{e}_m(-k) = (e_m)^\vee(k), \text{ or } F_m = (e_m)^\vee.$$

Therefore by Fourier inversion,

$$\hat{F}_m = (e_m^\vee)\hat{} = e_m.$$

<u># 2.3.5</u>: Let $u = (1, i, -1, -i)$, $v = (1, -1, 1, -1)$, and $z = (1, 1, i, -1, -1, 1, -i, -1)$.

i. Compute $\hat{u}$ and $\hat{v}$. (Suggestion: use Exercise 2.3.3.)

<u>Solution:</u> Note that for $N = 4$, $F_1(n) = \frac{1}{4}e^{2\pi in/4} = \frac{1}{4}\left(e^{i\pi/2}\right)^n = \frac{1}{4}i^n$. Hence

$$4F_1 = (i^0, i^1, i^2, i^3) = (1, i, -1, -i) = u.$$

By Exercise 2.3.3, $\hat{u} = 4\hat{F}_1 = 4e_1 = (0, 4, 0, 0)$. Similarly, $F_2(n) = \frac{1}{4}e^{2\pi i2n/4} = \frac{1}{4}\left(e^{i\pi}\right)^n = \frac{1}{4}(-1)^n$. Hence

$$4F_2 = ((-1)^0, (-1)^1, (-1)^2, (-1)^3) = (1, -1, 1, -1) = v.$$

By Exercise 2.3.3, $\hat{v} = 4\hat{F}_2 = 4e_2 = (0, 0, 4, 0)$.

ii. Compute $\hat{z}$.

<u>Solution:</u> By equation (2.48),
$\hat{z}(0) = \hat{u}(0) + \hat{v}(0) = 0 + 0 = 0$,
$\hat{z}(1) = \hat{u}(1) + e^{-2\pi i\cdot 1/8}\hat{v}(1) = 4 + e^{-2\pi i\cdot 1/8}0 = 4$,
$\hat{z}(2) = \hat{u}(2) + e^{-2\pi i\cdot 2/8}\hat{v}(2) = 0 + e^{-\pi i\cdot 1/2}4 = -4i$, and
$\hat{z}(3) = \hat{u}(3) + e^{-2\pi i\cdot 3/8}\hat{v}(3) = 0 + e^{-2\pi i\cdot 3/8}0 = 0$.
By equation (2.49),
$\hat{z}(4) = \hat{z}(0 + 4) = \hat{u}(0) - \hat{v}(0) = 0 - 0 = 0$,
$\hat{z}(5) = \hat{z}(1 + 4) = \hat{u}(1) - e^{-2\pi i\cdot 1/8}\hat{v}(1) = 4 - e^{-2\pi i\cdot 1/8}0 = 4$,
$\hat{z}(6) = \hat{z}(2 + 4) = \hat{u}(2) - e^{-2\pi i\cdot 2/8}\hat{v}(2) = 0 - e^{-\pi i\cdot 1/2}4 = 4i$, and
$\hat{z}(7) = \hat{z}(3 + 4) = \hat{u}(3) - e^{-2\pi i\cdot 3/8}\hat{v}(3) = 0 - e^{-2\pi i\cdot 3/8}0 = 0$.
Hence $\hat{z} = (0, 4, -4i, 0, 0, 4, 4i, 0)$.

<u># 2.3.6</u>: Suppose $u = (a, b, c, d)$, $v = (\alpha, \beta, \gamma, \delta)$, and $z = (a, \alpha, b, \beta, c, \gamma, d, \delta)$. If

$$\hat{u} = (2, i, -1, 0) \text{ and } \hat{v} = (3, -2, 0, 4i),$$

find $\hat{z}$.

<u>Solution:</u> By equation (2.48),
$\hat{z}(0) = \hat{u}(0) + \hat{v}(0) = 2 + 3 = 5$,
$\hat{z}(1) = \hat{u}(1) + e^{-2\pi i\cdot 1/8}\hat{v}(1) = i + e^{-2\pi i\cdot 1/8}(-2) = i + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)(-2) = (1 + \sqrt{2})i$, since $e^{-2\pi i\cdot 1/8} = e^{-\pi i\cdot/4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$,
$\hat{z}(2) = \hat{u}(2) + e^{-2\pi i\cdot 2/8}\hat{v}(2) = -1 + e^{-\pi i\cdot 1/2}0 = -1$, and
$\hat{z}(3) = \hat{u}(3) + e^{-2\pi i\cdot 3/8}\hat{v}(3) = 0 + e^{-2\pi i\cdot 3/8}4i = (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)(4i) = 2\sqrt{2} - 2\sqrt{2}i$, since $e^{-2\pi i\cdot 3/8} = e^{-3\pi i\cdot/4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.
By equation (2.49),
$\hat{z}(4) = \hat{z}(0 + 4) = \hat{u}(0) - \hat{v}(0) = 2 - 3 = -1$,
$\hat{z}(5) = \hat{z}(1 + 4) = \hat{u}(1) - e^{-2\pi i\cdot 1/8}\hat{v}(1) = i - e^{-2\pi i\cdot 1/8}(-2) = i - (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)(-2) = \sqrt{2} + (1 - \sqrt{2})i$
$\hat{z}(6) = \hat{z}(2 + 4) = \hat{u}(2) - e^{-2\pi i\cdot 2/8}\hat{v}(2) = -1 - e^{-\pi i\cdot 1/2}0 = -1$, and
$\hat{z}(7) = \hat{z}(3 + 4) = \hat{u}(3) - e^{-2\pi i\cdot 3/8}\hat{v}(3) = 0 - e^{-2\pi i\cdot 3/8}4i = -(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)(4i) = -2\sqrt{2} + 2\sqrt{2}i$.
Hence

$$\hat{z} = (5, -\sqrt{2} + (1 + \sqrt{2})i, -1, 2\sqrt{2} - 2\sqrt{2}i, -1, \sqrt{2} + (1 - \sqrt{2})i), -1, -2\sqrt{2} + 2\sqrt{2}i).$$