Contents

§1.5 # 1.5.4, 1.5.5, 1.5.6, 1.5.8, 1.5.9, 1.5.11

1.5.4: Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

i. Regarding A as a matrix over $\mathbf{R}$, show that A has no real eigenvalues.

Solution: The eigenvalues of A are the roots of $\det(\lambda I - A)$. Note that

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix}.$$

Hence $\det(\lambda I - A) = \lambda^2 + 1$, which has no real roots.

ii. Regarding A as a matrix over $\mathbb{C}$, find the complex eigenvalues of A and a corresponding eigenvector for each eigenvalue.

Solution: Since $\det(\lambda I - A) = \lambda^2 + 1 = (\lambda + i)(\lambda - i)$, the eigenvalues of A are $+i$ and $-i$.

For $\lambda = +i$, we consider $(\lambda I - A)v = 0$, that is, $(iI - A)v = 0$, or

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the two equations $ia + b = 0$ and $-a + ib = 0$. Multiplying the first equation by i gives the second equation, so these yield only one constraint, namely $b = -ia$. Thus E_i, the eigenspace for $\lambda = i$, consists of all vectors of the form $\begin{bmatrix} a \\ -ia \end{bmatrix}$. In particular, E_i is spanned by the eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

For $\lambda = -i$, when we consider $(\lambda I - A)v = (-iI - A)v = 0$, we obtain

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the two equations $-ia + b = 0$ and $-a - ib = 0$. Multiplying the first equation by $-i$ gives the second equation, so these yield only one constraint, namely $b = ia$. Thus E_{-i}, the eigenspace for $\lambda = -i$, consists of all vectors of the form $\begin{bmatrix} a \\ ia \end{bmatrix}$. In particular, E_i is spanned by the eigenvector $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

iii. Find an invertible matrix P over $\mathbb{C}$ and a diagonal matrix D over $\mathbf{C}$ such that $P^{-1}AP = D$.

Solution: By Lemma 1.74, we can take

$$D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}.$$

iv. Use part iii to calculate A^{99}.

Solution: Note that (by the formula for the inverse of a 2×2 matrix at the bottom of p. 46),

$$P^{-1} = \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}.$$

Since $A = PDP^{-1}$, we have (as on p. 72) $A^k = PD^kP^{-1}$. Note that

$$D^{99} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^{99} = \begin{bmatrix} i^{99} & 0 \\ 0 & (-i)^{99} \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

since $i^{99} = i^{96}i^3 = \left(i^4\right)^{24} i^3 = 1^{24}(-i) = -i$ and hence $(-i)^{99} = (-1)^{99}i^{99} = -(-i) = i$. Therefore

$$\begin{aligned} A^{99} = PD^{99}P^{-1} &= \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \\ &= \frac{1}{2i}\begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned}$$

v. Check that $A^2 = -I$. Use this to compute A^{99}, and compare with the answer in part iv.

Solution: We have

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I,$$

hence

$$A^{99} = \left(A^2\right)^{49} A = (-I)^{49}A = (-1)^{49}A = -A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

as in part iv.

1.5.5: Define $x_0 = 1$ and $x_1 = -1$. For $n \geq 0$, inductively define $x_{n+2} = x_{n+1} + 6x_n$. Find a formula for x_n.

Solution: For $n \geq 0$, define $u_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$. Then $u_0 = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and

$$u_{n+1} = \begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = Au_n, \text{ for } A = \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix}.$$

Hence (as on p. 74), $u_n = A^n u_0$. We diagonalize A in order to compute A^n. To find the eigenvalues of A, we note that

$$\det\,(\lambda I - A) = \det\begin{bmatrix} \lambda - 1 & -6 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - \lambda - 6 = (\lambda + 2)(\lambda - 3).$$

Hence the eigenvalues are $\lambda = -2$ and $\lambda = 3$.

To find an eigenvector v for $\lambda = -2$, we consider

$$(-2I - A)v = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields the equation $a + 2b = 0$. We obtain the eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

For $\lambda = 3$, we consider

$$(3I - A)v = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields the equation $-a + 3b = 0$. We obtain the eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Hence (as on p. 72) $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \frac{-1}{5} \begin{bmatrix} 1 & -3 \\ -1 & -2 \end{bmatrix}.$$

Therefore $A^n = PD^nP^{-1}$, hence $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = u_n = A^n u_0 = PD^nP^{-1}u_0$

$$= \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^n & 0 \\ 0 & 3^n \end{bmatrix} \frac{-1}{5} \begin{bmatrix} 1 & -3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{-1}{5} \begin{bmatrix} (-2)^{n+1} & 3^{n+1} \\ (-2)^n & 3^n \end{bmatrix} \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4(-2)^{n+1} + 3^{n+1} \\ 4(-2)^n + 3^n \end{bmatrix}.$$

The result is that

$$x_n = \frac{4}{5}(-2)^n + \frac{1}{5} \cdot 3^n.$$

1.5.6: The *Fibonacci sequence* is defined inductively by $x_0 = 0$, $x_1 = 1$, and for $n \geq 0$, $x_{n+2} = x_{n+1} + x_n$. Find a formula for the n^{th} *Fibonacci number* x_n.

Solution: For $n \geq 0$, define $u_n = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$. Then $u_0 = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and

$$u_{n+1} = \begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = Au_n, \quad \text{for } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence (as on p. 74), $u_n = A^n u_0$. We diagonalize A in order to compute A^n. To find the eigenvalues of A, we note that

$$\det\,(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - \lambda - 1.$$

By the quadratic formula, the eigenvalues of A are $\lambda = \frac{1 \pm \sqrt{5}}{2}$.

To find an eigenvector v for $\lambda = \frac{1+\sqrt{5}}{2}$, we consider

$$(\lambda I - A)v = \begin{bmatrix} \frac{1+\sqrt{5}}{2} - 1 & -1 \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{-1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields the equations $\frac{-1+\sqrt{5}}{2}a - b = 0$ and $-a + \frac{1+\sqrt{5}}{2}b = 0$. Multiplying the first by $-\frac{1+\sqrt{5}}{2}$ yields the second, so these give only one constraint: $b = \frac{-1+\sqrt{5}}{2}a$. Letting $a = 1$, we obtain the eigenvector $\begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix}$.

For $\lambda = \frac{1-\sqrt{5}}{2}$, we get

$$(\lambda I - A)v = \begin{bmatrix} \frac{1-\sqrt{5}}{2} - 1 & -1 \\ -1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields the equations $-\frac{1+\sqrt{5}}{2}a - b = 0$ and $-a + \frac{1-\sqrt{5}}{2}b = 0$. Multiplying the first by $-\frac{1-\sqrt{5}}{2}$ yields the second, so these give only one constraint: $b = -\frac{1+\sqrt{5}}{2}a$. Letting $a = 1$, we obtain the eigenvector $\begin{bmatrix} 1 \\ -\frac{1+\sqrt{5}}{2} \end{bmatrix}$.

Hence (as on p. 72) $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \end{bmatrix}, \quad D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}, \quad \text{and } P^{-1} = \frac{-1}{\sqrt{5}} \begin{bmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{bmatrix}.$$

Therefore $A^n = PD^nP^{-1}$, hence

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = u_n = A^n u_0 = PD^nP^{-1}u_0$$

$$= \begin{bmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \frac{-1}{\sqrt{5}} \begin{bmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{-1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} -\frac{1+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$= \frac{-1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} -\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \\ \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix}$$

$$= \frac{-1}{\sqrt{5}} \begin{bmatrix} -\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\ \frac{1-\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2^{n+1}\sqrt{5}} \left[(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\right] \\ \frac{1}{2^n\sqrt{5}} \left[(1+\sqrt{5})^n - (1-\sqrt{5})^n\right] \end{bmatrix},$$

using $\frac{1-\sqrt{5}}{2}\frac{1+\sqrt{5}}{2} = -1$ twice to simplify the bottom row. The result is that

$$x_n = \frac{1}{2^n\sqrt{5}}\left[(1+\sqrt{5})^n - (1-\sqrt{5})^n\right].$$

1.5.8: Let

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

Show that 3 is a eigenvalue of A which has algebraic multiplicity 2 but geometric multiplicity 1. Conclude that A does not have a basis of eigenvectors and hence is not diagonalizable.

Solution: Note that

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda - 3 & -1 \\ 0 & \lambda - 3 \end{bmatrix}.$$

Hence $\det \lambda I - A = (\lambda - 3)^2$. Therefore 3 is an eigenvalue with algebraic multiplicity 2. To find the eigenspace E_3 corresponding to the eigenvalue 3, we solve $(\lambda I - A)u = (3I - A)u = 0$, or

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which gives only one constraint, namely $b = 0$. Therefore E_3 consists of all vectors of the form $\begin{bmatrix} a \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, with a a scalar. In other words, E_3 is spanned by the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So E_3 is 1-dimensional, so $\lambda = 3$ has geometric multiplicity 1. Since this is less than the algebraic multiplicity, there is no basis of eigenvectors of A, and so A is not diagonalizable.

1.5.9: Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 1 \\ 3 & 0 & 3 \end{bmatrix}.$$

i. Find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$.

Solution: Note that $\lambda I - A = \begin{bmatrix} \lambda-2 & 0 & 0 \\ -3 & \lambda-2 & -1 \\ -3 & 0 & \lambda-3 \end{bmatrix}$. Hence Det $(\lambda I - A) = (\lambda-2)(\lambda-2)(\lambda-3) = (\lambda-2)^2(\lambda-3)$. So A has eigenvalue 2 with algebraic multiplicity 2 and eigenvalue 3 with algebraic multiplicity 1.

For $\lambda = 2$, we consider

$$(2I - A)v = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & -1 \\ -3 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which gives the equation $-3a - c = 0$. Thus $c = -3a$, so the most general form of an eigenvector corresponding to $\lambda = 2$ is $\begin{bmatrix} a \\ b \\ -3a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Hence we can take $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as eigenvectors forming a basis for the eigenspace corresponding to $\lambda = 2$.

For $\lambda = 3$, we consider

$$(3I - A)v = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & -1 \\ -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which gives the equations $a = 0$ and then $b - c = 0$. Hence we can take $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ as an eigenvector corresponding to $\lambda = 3$.

Therefore we let

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -3 & 0 & 1 \end{bmatrix}.$$

ii. Find a matrix B such that $B^2 = A$.

Solution: Since $P^{-1}AP = D$, we have $A = PDP^{-1}$. Also the diagonal matrix $C = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$ satisfies $C^2 = D$. Thus if we let $B = PCP^{-1}$, we have $B^2 = PCP^{-1}PCP^{-1} = PC^2P^{-1} = PDP^{-1} = A$. So to find B we need to compute P^{-1} and then PCP^{-1}. To find P^{-1}, we use the Gaussian elimination technique for matrix inversion as follows:

$$[P|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & -1 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right] = [I|P^{-1}],$$

where at the first step we added three times the first row to the third, and at the second step we subtracted the third row from the second. Hence $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix}$. Therefore

$$B = PCP^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{3} \\ -3\sqrt{2} & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -3\sqrt{2}+3\sqrt{3} & \sqrt{2} & -\sqrt{2}+\sqrt{3} \\ -3\sqrt{2}+3\sqrt{3} & 0 & \sqrt{3} \end{bmatrix}.$$

<u># 1.5.11</u>: Find the general solution to the system of differential equations

$$y_1' = 8y_1 - 15y_2$$

$$y_2' = 2y_1 - 3y_2.$$

<u>Solution:</u> We write

$$y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 8 & -15 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ay, \text{ for } A = \begin{bmatrix} 8 & -15 \\ 2 & -3 \end{bmatrix}.$$

To diagonalize A, we compute

$$\det\ (\lambda I - A) = \det \begin{bmatrix} \lambda - 8 & 15 \\ -2 & \lambda + 3 \end{bmatrix} = \lambda^2 - 5\lambda - 24 + 30 = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2).$$

Hence the eigenvalues of A are $\lambda = 2$ and $\lambda = 3$.

To find an eigenvector corresponding to $\lambda = 2$, we solve $(2I - A)u = 0$, or

$$\begin{bmatrix} -6 & 15 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields the one constraint $-2a + 5b = 0$. So we can take $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ as an eigenvector for $\lambda = 2$.

For $\lambda = 3$, solving $(3I - A)u = 0$ gives

$$\begin{bmatrix} -5 & 15 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or $-a + 3b = 0$. So we can take $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ as an eigenvector for $\lambda = 3$.

As on p. 72, we obtain $P^{-1}AP = D$, or equivalently $A = PDP^{-1}$, for

$$P = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \text{and } P^{-1} = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}.$$

The equation $y' = Ay$ is the same as $y' = PDP^{-1}y$, which is equivalent to $P^{-1}y' = DP^{-1}y$. Since multiplying by constants commutes with differentiation, this is the same as $(P^{-1}y)' = DP^{-1}y$. Let $z = P^{-1}y$. Then the equation is $z' = Dz$, or

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

which is just the pair of uncoupled equations $z_1' = 2z_1$ and $z_2' = 3z_2$. By basic o.d.e., the solutions are $z_1(t) = C_1e^{2t}$ and $z_2(t) = C_2e^{3t}$. Since $y = Pz$, we get

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} C_1e^{2t} \\ C_2e^{3t} \end{bmatrix} = \begin{bmatrix} 5C_1e^{2t} + 3C_2e^{3t} \\ 2C_1e^{2t} + C_2e^{3t} \end{bmatrix}.$$

In other words, $y_1(t) = 5C_1e^{2t} + 3C_2e^{3t}$ and $y_2(t) = 2C_1e^{2t} + C_2e^{3t}$.

§1.6 # 1.6.6, 1.6.19

1.6.6: i. Let V be a complex inner product space and $\| \cdot \|$ a norm obtained from the inner product as in Definition 1.90. Prove the *parallelogram identity*:

$$2 \| z \|^2 + 2 \| w \|^2 = \| z + w \|^2 + \| z - w \|^2,$$

for every $z, w \in V$. To see why this is called the parallelogram identity, draw two nonzero vectors $z, w \in \mathbb{R}^2$ with a common base point and consider the parallelogram with sides z and w. Show that the sums of the squares of the lengths of the two diagonals equals the sums of the squares of the lengths of the four sides.

Solution: We expand the right side by the definition and apply additivity (that is, (I1) in Definition 1.86 and equation (1.42)), and scalar homogeneity (I2 in Definition 1.86 and equation (1.43)):

$$\| z + w \|^2 + \| z - w \|^2 = \langle z + w, z + w \rangle + \langle z - w, z - w \rangle$$
$$= \langle z, z \rangle + \langle z, w \rangle + \langle w, z \rangle + \langle w, w \rangle + \langle z, z \rangle - \langle z, w \rangle - \langle w, z \rangle + \langle w, w \rangle$$
$$= 2\langle z, z \rangle + 2\langle w, w \rangle = 2 \| z \|^2 + 2 \| w \|^2,$$

as required.

ii. For $z \in \mathbb{C}^2$, define

$$\| z \|_1 = |z_1| + |z_2|,$$

where z_1 and z_2 are the components of z. Prove that $\| \cdot \|_1$ is a norm on $\mathbb{C}^2$. This is called the ℓ^1 norm.

Solution: We check properties N1-N4 in the definition of a norm in Exercise 1.6.5.

N1: For any $z \in \mathbb{C}^2$, $\| z \|_1 = |z_1| + |z_2|$ is clearly a non-negative real number.

N2: We have $\| z \|_1 = |z_1| + |z_2| = 0$ if and only if $z_1 = 0$ and $z_2 = 0$, that is, if and only if $z = 0$.

N3: By definition, $\| \lambda z \|_1 = |\lambda z_1| + |\lambda z_2| = |\lambda||z_1| + |\lambda||z_2| = |\lambda|(|z_1| + |z_2|) = |\lambda| \| z \|_1$.

N4: For any $z, w \in \mathbb{C}^2$, using the definition of the norm and the triangle inequality on $\mathbb{R}$ (Lemma 1.2) gives $\| z + w \|_1 = |z_1 + w_1| + |z_2 + w_2| \leq |z_1| + |w_1| + |z_2| + |w_2| = |z_1| + |z_2| + |w_1| + |w_2| = \| z \|_1 + \| w \|_1$.

iii. Find vectors $z, w \in \mathbb{C}^2$ such that

$$2 \| z \|_1^2 + 2 \| w \|_1^2 \neq \| z + w \|_1^2 + \| z - w \|_1^2 .$$

Deduce that $\| \cdot \|_1$ is a norm that does not come from an inner product.

Solution: Almost any choice will do. For example, take $z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $z + w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $z - w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then $\| z \|_1 = 1$, $\| w \|_1 = 1$, $\| z + w \|_1 = 2$, and $\| z - w \|_1 = 2$. Hence

$$2 \| z \|_1^2 + 2 \| w \|_1^2 = 2 \cdot 1^2 + 2 \cdot 1^2 = 4,$$

while

$$\| z+w \|_1^2 + \| z-w \|_1^2 = 2^2 + 2^2 = 4 + 4 = 8.$$

Thus the ℓ^1 norm does not satisfy the parallelogram law, which is satisfied by every norm coming from an inner product as in Definition 1.90. So $\| \cdot \|_1$ is a norm that does not come from an inner product.

1.6.19: Let A be an $n \times n$ matrix over $\mathbb{C}$. Prove that A is unitary if and only if A is normal and all eigenvalues of A have magnitude 1.

Solution: ($\Longrightarrow$) Suppose A is unitary. Then by definition $A^*A = I = AA^*$, so A is normal. Suppose λ is an eigenvalue of A, and let v be an eigenvector for λ, so that $v \neq 0$ and $Av = \lambda v$. By Lemma 1.105 part v,

$$\|v\| = \|Av\| = \|\lambda v\| = |\lambda|\|v\|.$$

Since $\|v\| \neq 0$ (because $v \neq 0$), we get $|\lambda| = 1$.

($\Longleftarrow$) Suppose A is normal and all the eigenvalues of A have magnitude 1. By Theorem 1.109, there exist D diagonal and U unitary such that $A = U^*DU$. Since A is similar to D, they have the same eigenvalues (Corollary 1.68), so the diagonal entries $\lambda_1, \lambda_2, ..., \lambda_n$ of D (which are the eigenvalues of D, since D is diagonal) have magnitude 1. Then D^* has diagonal entries $\overline{\lambda}_1, \overline{\lambda}_2, ..., \overline{\lambda}_n$. Hence

$$D^*D = \begin{bmatrix} \overline{\lambda}_1 & 0 & \cdot & \cdot & 0 \\ 0 & \overline{\lambda}_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \overline{\lambda}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdot & \cdot & 0 \\ 0 & \lambda_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} \overline{\lambda}_1\lambda_1 & 0 & \cdot & \cdot & 0 \\ 0 & \overline{\lambda}_2\lambda_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \overline{\lambda}_n\lambda_n \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 & 0 & \cdot & \cdot & 0 \\ 0 & |\lambda_2|^2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & |\lambda_n|^2 \end{bmatrix} = I, .$$

because $|\lambda_j| = 1$ for all j. By Exercise 1.6.10 i, $A^* = (U^*DU)^* = U^*(U^*D)^* = U^*D^*(U^*)^* = U^*D^*U$, since $(U^*)^* = U$. Hence

$$A^*A = U^*D^*UU^*DU = U^*D^*DU = U^*IU = U^*U = I.$$

Thus A is invertible and $A^* = A^{-1}$ (Exercise 1.4.12 ii). That is, A is unitary.