

Contents

1 Chapter 1 Solutions	2
§1.2 # 1.2.11 i, iii, v, vi, 1.2.12 i, ii, iii, 1.2.13, 1.2.15	2
§1.3 # 1.3.9	4
§1.4 # 1.4.5, 1.4.7, 1.4.13	4

Chapter 1

Chapter 1 Solutions

1.1.6: Prove Lemma 1.8. Suggestion: Do not write it out in terms of the real and imaginary parts. Instead, prove that

$$|z + w|^2 = (z + w)(\overline{z + w}) = |z|^2 + 2\operatorname{Re}(\overline{z}w) + |w|^2,$$

and use Lemma 1.7.

Solution: First note that

$$\operatorname{Re}(\overline{z}w) = \frac{1}{2}(\overline{z}w + \overline{\overline{z}w}) = \frac{1}{2}(\overline{z}w + \overline{\overline{z}}\overline{w}) = \frac{1}{2}(\overline{z}w + z\overline{w}) = \frac{1}{2}(z\overline{w} + w\overline{z}),$$

using Lemma 1.7. Then using $\operatorname{Re}(\overline{z}w) \leq |\overline{z}w| = |\overline{z}||w| = |z||w|$, we obtain

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + 2\operatorname{Re}(\overline{z}w) + |w|^2 \leq |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2. \end{aligned}$$

That is, $|z + w|^2 \leq (|z| + |w|)^2$. Taking the square root of both sides gives $|z + w| \leq |z| + |w|$.

§1.2 # 1.2.11 i, iii, v, vi, 1.2.12 i, ii, iii, 1.2.13, 1.2.15

1.2.11 i, iii, v, vi: Write each of the following complex numbers in the form $a + ib$, with $a, b \in \mathbb{R}$, where your answer is stated without using the trigonometric functions:

i. $e^{3i\pi/2}$ iii. $e^{2\pi i/3}$ v. $5e^{-5i\pi/6}$ vi. $2e^{-32\pi i/3}$.

Solution: i. $e^{3i\pi/2} = \cos(3\pi/2) + i \sin(3\pi/2) = 0 + i(-1) = -i$

iii. $e^{2\pi i/3} = \cos(2\pi/3) + i \sin(2\pi/3) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

v. $5e^{-5i\pi/6} = 5(\cos(-5\pi/6) + i \sin(-5\pi/6)) = -5\frac{\sqrt{3}}{2} - 5\frac{1}{2}i = -5\frac{\sqrt{3}}{2} - \frac{5}{2}i$

vi. $2e^{-32i\pi/3} = 2e^{-36i\pi/3}e^{4i\pi/3} = 2e^{-12i\pi}e^{4i\pi/3} = 2e^{4i\pi/3} = 2(\cos(4\pi/3) + i \sin(4\pi/3)) = 2(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) = -1 - \sqrt{3}i$

1.2.12 i, ii, iii: Express the following complex numbers in polar form $re^{i\theta}$, with $r > 0$ and $0 \leq \theta < 2\pi$:

i. $4 + 4i$ ii. $2\sqrt{3} - 2i$ iii. $-3 + 3\sqrt{3}i$

Solution: i. Since $|4 + 4i| = \sqrt{16 + 16} = 4\sqrt{2}$, we write

$$4 + 4i = 4\sqrt{2} \left(\frac{4}{4\sqrt{2}} + \frac{4}{4\sqrt{2}}i \right) = 4\sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) = 4\sqrt{2}(\cos(\pi/4) + i \sin(\pi/4)) = 4\sqrt{2}e^{i\pi/4}.$$

ii. Since $|2\sqrt{3} - 2i| = \sqrt{12 + 4} = 4$, we write

$$2\sqrt{3} - 2i = 4 \left(\frac{2\sqrt{3}}{4} - \frac{2}{4}i \right) = 4 \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = 4(\cos(11\pi/6) + i \sin(11\pi/6)) = 4e^{i11\pi/6}.$$

iii. Since $|-3 + 3\sqrt{3}i| = \sqrt{9 + 27} = 6$, we write

$$-3 + 3\sqrt{3}i = 6 \left(\frac{-3}{6} + \frac{3\sqrt{3}}{6}i \right) = 6 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = 6(\cos(2\pi/3) + i \sin(2\pi/3)) = 6e^{i2\pi/3}.$$

1.2.13: Find $(\sqrt{3} + i)^{101}$. Write your answer in the form $a + ib$, with $a, b \in \mathbb{R}$, without using the trigonometric functions.

Solution: Since $|\sqrt{3} + i| = \sqrt{3 + 1} = 2$, we get

$$\sqrt{3} + i = 2 \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = 2(\cos(\pi/6) + i \sin \pi/6) = 2e^{i\pi/6}.$$

Therefore

$$\begin{aligned} (\sqrt{3} + i)^{101} &= 2^{101} \left(e^{i\pi/6} \right)^{101} = 2^{101} e^{i101\pi/6} = 2^{101} e^{i(16\pi + 5\pi/6)} = 2^{101} e^{i16\pi} e^{i5\pi/6} = 2^{101} e^{i5\pi/6} \\ &= 2^{101} (\cos(5\pi/6) + i \sin(5\pi/6)) = 2^{101} \left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i \right) = -2^{100}\sqrt{3} + 2^{100}i. \end{aligned}$$

1.2.15: Using a calculator, find all 4th roots of $3 + 4i$ to several decimal places accuracy.

Solution: Since $|3 + 4i| = \sqrt{9 + 16} = 5$, we have $3 + 4i = 5 \left(\frac{3}{5} + i\frac{4}{5} \right) = 5e^{i \arctan(4/3)}$, since $\cos \theta = 3/5$ and $\sin \theta = 4/5$ implies that $\tan \theta = 4/3$. Let $\theta = \arctan(4/3)$. Then

$$3 + 4i = 5e^{i\theta} = 5e^{i(\theta+2\pi)} = 5e^{i(\theta+4\pi)} = 5e^{i(\theta+6\pi)}.$$

Hence the fourth roots of $3 + 4i$ are z_1, z_2, z_3 and z_4 , where

$$z_1 = 5^{1/4} e^{i\theta/4}, \quad z_2 = 5^{1/4} e^{i(\theta+2\pi)/4} = 5^{1/4} e^{i\theta/4} e^{i\pi/2} = 5^{1/4} e^{i\theta/4} i = iz_1,$$

$$z_3 = 5^{1/4} e^{i(\theta+4\pi)/4} = 5^{1/4} e^{i\theta/4} e^{i\pi} = -5^{1/4} e^{i\theta/4} = -z_1,$$

and

$$z_4 = 5^{1/4} e^{i(\theta+6\pi)/4} = 5^{1/4} e^{i\theta/4} e^{i3\pi/2} = -5^{1/4} e^{i\theta/4} i = -iz_1.$$

From a calculator we get $\theta \approx .9272952$, hence $\theta/4 \approx .2318238$. Therefore

$$e^{i\theta/4} \approx \cos(.2318238) + i \sin .2318238 \approx .9732489 + .2297529 i.$$

Since $5^{1/4} \approx 1.4953488$, we get $5^{1/4} e^{i\theta/4} \approx 1.4953488(.9732489 + .2297529 i) \approx 1.4553466 + .3435607 i$. Hence

$$z_1 = 5^{1/4} e^{i\theta/4} \approx 1.4553466 + .3435607 i, \quad z_2 = iz_1 \approx -.3435607 + 1.4553466 i,$$

$$z_3 = -z_1 \approx -1.4553466 - .3435607 i, \quad z_4 = -iz_1 \approx .3435607 - 1.4553466 i.$$

§1.3 # 1.3.9

1.3.9: Suppose u, v_1, v_2, \dots, v_n are vectors in some vector space.

i. If $u \in \text{span} \{v_1, v_2, \dots, v_n\}$, prove that u, v_1, v_2, \dots, v_n are linearly dependent.

Solution: Since $u \in \text{span} \{v_1, v_2, \dots, v_n\}$, there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $u = \alpha_1 v_1 + \dots + \alpha_n v_n$. Therefore

$$-1 \cdot u + \alpha_1 v_1 + \dots + \alpha_n v_n = 0.$$

This is a non-trivial linear combination (since the first coefficient is nonzero) of u, v_1, \dots, v_n which is 0, hence u, v_1, \dots, v_n are linearly dependent.

ii. Suppose v_1, v_2, \dots, v_n are linearly independent. If u, v_1, v_2, \dots, v_n are linearly dependent, prove that $u \in \text{span} \{v_1, v_2, \dots, v_n\}$.

Solution: Since u, v_1, v_2, \dots, v_n are linearly dependent, there exist scalars $\alpha, \alpha_1, \dots, \alpha_n$ which are not all 0 such that

$$\alpha u + \alpha_1 v_1 + \dots + \alpha_n v_n = 0.$$

If $\alpha = 0$, then at least one of $\alpha_1, \dots, \alpha_n$ are nonzero and $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$. This contradicts the linear independence of v_1, v_2, \dots, v_n . Hence $\alpha \neq 0$. Therefore we can divide by α and solve for u to obtain

$$u = -\frac{\alpha_1}{\alpha} v_1 - \frac{\alpha_2}{\alpha} v_2 - \dots - \frac{\alpha_n}{\alpha} v_n.$$

Hence $u \in \text{span} \{v_1, v_2, \dots, v_n\}$.

§1.4 # 1.4.5, 1.4.7, 1.4.13

1.4.5: Define a basis R for \mathbb{R}^3 by

$$R = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\},$$

and a basis S for \mathbb{R}^2 by

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}.$$

Define $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y - z \\ 2x - y + 3z \end{bmatrix}.$$

i. Find the matrix A which represents T with respect to R and S .

Solution: By Lemma 1.49, we need to solve

$$T \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = a_{11} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_{21} \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \text{ or, equivalently } \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$T \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 7 \end{bmatrix} = a_{12} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_{22} \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \text{ or, equivalently } \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix},$$

and

$$T\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \end{bmatrix} = a_{13} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_{23} \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \text{ or, equivalently } \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix},$$

where we used the definition of T at the first step each time. Now

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix},$$

hence

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \end{bmatrix}$$

and

$$\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ -6 \end{bmatrix}.$$

Thus

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 8 & -9 & 17 \\ -3 & 5 & -6 \end{bmatrix}.$$

ii. Suppose $u \in \mathbb{R}^3$ and

$$[u]_R = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Find $[T(u)]_S$.

Solution:

$$[T(u)]_S = A[u]_R = \begin{bmatrix} 8 & -9 & 17 \\ -3 & 5 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix}.$$

iii. For u as in part ii, find the Euclidean coordinates of u .

Solution:

$$u = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}.$$

iv. Find the Euclidean coordinates of $T(u)$ two ways: using part iii and the definition of T , and using part ii and the definition of S .

Solution: Method 1: by part iii,

$$T(u) = \begin{bmatrix} -2 - 1 - 0 \\ 2(-2) - (-1) + 3(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

Method 2: by part ii,

$$T(u) = -9 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

1.4.7: Let U and V be vector spaces and $T : U \rightarrow V$ a linear transformation. Prove that T is $1 - 1$ if and only if $\ker T = \{0\}$.

Solution: Suppose T is $1 - 1$ and $u \in \ker T$. Since $T(0) = 0 = T(u)$, we obtain $u = 0$ by the definition of $1 - 1$. Hence $\ker T = \{0\}$.

Now suppose $\ker T = \{0\}$. Suppose $T(u) = T(v)$. Since T is linear, we have $T(u - v) = 0$, or $u - v \in \ker T = \{0\}$. Therefore $u - v = 0$, or $u = v$. This shows that T is 1-1.

1.4.13: Define bases R and S for \mathbb{R}^2 by

$$R = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\},$$

and

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}.$$

i. Find the R to S change of basis matrix A .

Solution: By Lemma 1.58, we need to solve

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = a_{11} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + a_{21} \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \text{ or, equivalently } \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = a_{12} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + a_{22} \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \text{ or, equivalently } \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Now

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix},$$

hence

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Thus

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 2 \end{bmatrix}.$$

ii. Suppose $x \in \mathbb{R}^2$ and

$$[x]_R = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find $[x]_S$.

Solution:

$$[x]_S = A[x]_R = \begin{bmatrix} -3 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

iii. Find the Euclidean coordinates of x two ways: using $[x]_R$ and the definition of R , and using $[x]_S$ and the definition of S .

Solution: First way:

$$x = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}.$$

Second way:

$$x = -5 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}.$$

iv. Find the S to R change of basis matrix directly, that is, by the method of Lemma 1.58. Then check that it is the inverse of the matrix found in part i.

Solution: For A found in part i, we get

$$A^{-1} = \frac{1}{-4} \begin{bmatrix} 2 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/4 \\ 1/2 & 3/4 \end{bmatrix}.$$

To find the S to R change of basis matrix B , we need to solve

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = b_{11} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_{21} \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \text{ or, equivalently } \begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = b_{12} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_{22} \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \text{ or, equivalently } \begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Now

$$\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 7 & -3 \\ -1 & 1 \end{bmatrix},$$

hence

$$\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 7 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 7 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 3/4 \end{bmatrix}.$$

Thus

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} -1/2 & -1/4 \\ 1/2 & 3/4 \end{bmatrix} = A^{-1}.$$