WEEK 8, PART II: EXCITATION

1. The Radio Frequency Field and Non-selective Excitation

We use the RF field to generate the excitation that was assumed in the previous section. What the RF field does is to essentially 'tip' the bulk magnetization from its equilibrium position, pointing up the z axis, down toward the xy plane. We assume for the moment that the magnetization was allowed to return to equilibrium after the previous excitation.

The standard form of $\boldsymbol{B}_{RF}$ is

$$\boldsymbol{B}_{RF}(t) = (\alpha(t)\cos\omega_0 t, -\alpha(t)\sin\omega_0 t, 0)^t, \tag{1.1}$$

where α is called the *pulse envelope function.* Thus, α indicates the strength of the RF field.

Recall the Bloch equation in the rotating reference frame

$$\frac{\partial \boldsymbol{m}(r,t)}{\partial t} = \boldsymbol{m}(r,t) \times \gamma\, \boldsymbol{B}_{\text{eff}}(r,t) - \frac{\boldsymbol{m_{xy}}(r,t)}{T_2(r)} - \frac{(m_z(r,t) - M_z^0(r,t))\boldsymbol{k}}{T_1(r)}.$$

In what follows we denote $\dfrac{\partial \boldsymbol{m}}{\partial t}$ by $\boldsymbol{m'}$. Since the excitation pulse is short compared to the relaxation terms, we can ignore the relaxation terms to get

$$\boldsymbol{m}'(r,t) = \boldsymbol{m}(r,t) \times \gamma\, \boldsymbol{B}_{\text{eff}}(r,t). \tag{1.2}$$

We will ignore the spacial coordinate r in this section, since we are discussing nonselective excitations in which location plays no role. When the background field takes the form $\boldsymbol{B}(t) = \boldsymbol{B}_0 + \boldsymbol{B}_{RF}(t)$, where $\boldsymbol{B}_{RF}(t)$ is given by equation (1.1), the field $\boldsymbol{B}_{\text{eff}}(t) = (\alpha(t), 0, 0)^t$. Then the solution to (1.2) is

$$\boldsymbol{m}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & \sin\theta(t) \\ 0 & -\sin\theta(t) & \cos\theta(t) \end{pmatrix} \boldsymbol{m}(0),$$

where

$$\theta(t) = \gamma \int_0^t \alpha(\tau)\, d\tau$$

is called the *flip* or *tip angle*.

Hence, by choosing a pulse length τ and a function α supported on $[0, \tau]$ such that

$$\gamma \int_0^\tau \alpha(t)dt = \frac{\pi}{2},$$

we can rotate the bulk magnetization from its equilibrium position down to the xy plane. This maximizes the motion of the bulk magnetization in free precession (that is, it rotates at the resonance frequency in the widest possible circle). Pulses are often named for the shaping function α; two of the most common are the box pulse and sinc pulse. We can similarly rotate the magnetization any angle we like; smaller flip-angles are sometimes used in fast imaging techniques for reasons we will describe later. 180° pulses are also common, although they have a different purpose from the smaller-angle pulses used in excitation. We will describe their use later when we discuss spin-echo imaging.

2. Slice Selection

A considerably more difficult problem than that of non-selective excitation described above is the problem of exciting the magnetization only in a thin cross-section, or 'slice', of the patient.

It is technologically not feasible to achieve selective excitation of a single slice by confining the effect of either the background or radio frequency fields to that slice. Instead, slice selection is achieved using the interaction between the RF field and a nonzero gradient field. The gradient subtly changes the resonance frequencies within the object, and a carefully designed pulse envelope function will excite magnetization with resonant frequencies only within a narrow band.

Suppose we would like to excite the magnetic moments in some two-dimensional plane $\boldsymbol{P} \subset \mathbb{R}^3$. This, of course, is not possible, so a more realistic goal would be to excite the spins in between the two parallel planes $\boldsymbol{P} \pm \boldsymbol{\Delta} r$, where $\boldsymbol{\Delta} r$ is some small vector normal to $\boldsymbol{P}$, that is, orthogonal to every vector in $\boldsymbol{P}$. In the simplest case, $\boldsymbol{P}$ is a horizontal slice and $\boldsymbol{\Delta} r = (0, 0, \Delta z)$ for some small scalar Δz.

For the moment, we ignore the radio frequency field and look at how to choose an appropriate gradient. Let the magnetic field take the form

$$\boldsymbol{B}(r,t) = \boldsymbol{B}_0 + \boldsymbol{G}(r),$$

where $\boldsymbol{B}_0 = (0, 0, b_0)$ and the effective gradient field is $\boldsymbol{G}(r) = (0, 0, r \cdot \boldsymbol{g})$ for a fixed gradient vector $\boldsymbol{g}$ (recall $r = (x, y, z) \in \mathbb{R}^3$). In the simplest case, $\boldsymbol{g}$ is the (constant) gradient G_z, so $r \cdot \boldsymbol{g} = z\,G_z$. In general, $\boldsymbol{g}$ can be pointed in any direction. Suppose that $\boldsymbol{g}$ is normal to $\boldsymbol{P}$, that is, orthogonal to every vector in $\boldsymbol{P}$. Then the level sets of $G(r)$, and hence, those of $\boldsymbol{B}_0 + \boldsymbol{G}(r)$, will be parallel to $\boldsymbol{P}$. Let $r_0 \in \boldsymbol{P}$. We may re-define $\boldsymbol{B}_0$ and $\boldsymbol{G}(r)$ so that $G(r_0) = 0$ by taking $\boldsymbol{B}_0 = \boldsymbol{B}_0 + \boldsymbol{G}(r_0)$ if necessary, and since $\boldsymbol{P}$ is a level set of $\boldsymbol{G}$, this means we may assume that $\boldsymbol{G}(r) = 0$ for every r in $\boldsymbol{P}$.

Define ω_0 as $\omega_0 = \gamma b_0$. Since we normalized so that $\boldsymbol{G} = 0$ on P, ω_0 is the resonant frequency at points $r \in P$. For $r \notin P$, the resonant frequency, given by

$$\omega(r) = \omega_0 + \gamma \boldsymbol{g} \cdot (r - \boldsymbol{P}), \tag{2.1}$$

will differ from ω_0 due to the alteration in the strength of the field. Let

$$\Delta\omega = \gamma\,|\boldsymbol{g} \cdot \boldsymbol{\Delta} r|. \tag{2.2}$$

So in the simplest case, the frequence shift in (2.2) becomes $\Delta\omega = \gamma\,\Delta z\,G_z$. Then the problem of exciting spins for the points between the planes $\boldsymbol{P} \pm \boldsymbol{\Delta} r$ is equivalent to the problem of exciting spins whose resonant frequencies lie between $\omega_0 - \Delta\omega$ and $\omega_0 + \Delta\omega$.

In the rotating frame (rotating at frequency ω_0), the effective magnetic field is

$$\boldsymbol{B}_{\text{eff}}(r,t) = (0, 0, \boldsymbol{g} \cdot (r - \boldsymbol{P})). \tag{2.3}$$

Here, $r - \boldsymbol{P}$ is the difference between r and $\boldsymbol{P}$, defined as the *argmin* of $\|r - p\|$ where p ranges over P, that is, the shortest vector that can be added to r in order to get something in P, which agrees with the intuitive geometric notion of difference between a point and a plane. At any particular point r, the magnetization will have frequency f in the rotating frame, where

$$f(r) = \gamma \boldsymbol{g} \cdot (r - \boldsymbol{P}) = \omega(r) - \omega_0.$$

Note that we use f, rather than ω, to emphasize that f is the frequency in the rotating, rather than laboratory, frame. We call $f(r)$ the *offset frequency*, and for the remainder of this section, we will write $\boldsymbol{m}(f,t)$ in place of $\boldsymbol{m}(r,t)$, under the tacit understanding that $f = f(r) = f(x,y,z)$ depends on the location. Using this notation, our goal is to find a radio frequency pulse such that $\boldsymbol{m}(f,\cdot)$ is excited for $f \in [-\Delta\omega, \Delta\omega]$ and not for other values of f.

Now suppose we turn on the radio frequency field at the resonant frequency ω_0 of points in P, that is, let

$$\boldsymbol{B}(r,t) = \boldsymbol{B}_0 + \boldsymbol{B}_{RF}(t) + \boldsymbol{G}(r,t),$$

where $\boldsymbol{B}_{RF}(t) = (\alpha(t)\cos\omega_0 t, -\alpha(t)\sin\omega_0 t, 0)$. In the rotating reference frame (rotating at frequency ω_0), this leads to an effective magnetic field of

$$\boldsymbol{B}_{\text{eff}}(r,t) = (\alpha(t), 0, \boldsymbol{g}\cdot(r-\boldsymbol{P})). \tag{2.4}$$

The Bloch equation in the rotating frame, ignoring relaxation terms, can now be written in terms of f and α as

$$\boldsymbol{m}'(f,t) = \begin{pmatrix} 0 & f & 0 \\ -f & 0 & \gamma\alpha(t) \\ 0 & -\gamma\alpha(t) & 0 \end{pmatrix} \boldsymbol{m}(f,0),$$

or in scalar form,

$$\begin{cases} m_x'(f,t) &= \quad f\, m_y(f,t), \\ m_y'(f,t) &= -\; f\, m_x(f,t) + \gamma\,\alpha(t)\, m_z(f,t), \\ m_z'(f,t) &= \quad \gamma\,\alpha(t)\, m_y(f,t). \end{cases} \tag{2.5}$$

Suppose that we want to create a flip angle of θ within the selected slice. Then we define an ideal slice-selection profile $\boldsymbol{p}$ by

$$\boldsymbol{p}\,(f) = \begin{cases} (0, M_z^0\cos\theta, M_z^0\,\sin\theta), & f\in[-\Delta\omega,\Delta\omega] \\ (0,0,M_z^0), & f\notin[-\Delta\omega,\Delta\omega]. \end{cases}$$

Our goal is then to find a pulse envelope function α (i.e., the strength of an RF pulse), which we will assume is nonzero only within the time interval $[0,\tau_p]$, such that, whenever the initial magnetization profile is

$$\boldsymbol{M}(f,0) = \boldsymbol{m}(f,0) = \boldsymbol{M}^0,$$

the magnetization profile at the end of the pulse comes as close as possible to satisfying

$$\boldsymbol{m}(f,\tau_p) = \boldsymbol{p}\,(f).$$

The problem is nonlinear, and, in general, there is no closed-form solution. We can, however, approximate with a linear model in the case when the flip angle is small by assuming that the z-coordinate of $\boldsymbol{m}$ does not change, and the solution turns out to be acceptable, although less than ideal and not used in practice, for flip angles of up to 90°.

Let M_z^0 be the magnitude of $\boldsymbol{M}^0$, which points in the z direction. Then we linearize the system (2.5) by assuming that $m_z \approx M_z^0$ for all t, and we try to create the desired transverse magnetization $\boldsymbol{M}_{xy}$ under this assumption. The simplified system of differential equations is

$$\begin{cases} m_x'(f,t) &= \quad f\, m_y(f,t), \\ m_y'(f,t) &= -\; f\, m_x(f,t) + \gamma\,\alpha(t)\, M_z^0, \\ m_z'(f,t) &= \quad 0. \end{cases} \tag{2.6}$$

We now convert the problem into one involving a single complex derivative by following the familiar convention of writing a point (x,y) in the first two coordinates as $x+iy$, and writing the vector m_x+im_y as m_{xy}. In this notation, we have

$$m_{xy}'(f,t) = -if\, m_{xy}(f,t) + i\gamma\,\alpha(t)\, M_z^0. \tag{2.7}$$

In this context, we are only concerned with the transverse part of the resulting profile, so that the target $m_{xy}(f,\tau_p)$ profile (in complex form) is

$$\boldsymbol{p}_{|xy}(f) = iM_z^0\,\cos\theta\,\chi_{[-1,1]}\left(\frac{f}{\Delta\omega}\right), \tag{2.8}$$

where $\chi_A(x)$ means $\chi(x)=1$ if $x\in A$ and $\chi(x)=0$ if $x\notin A$.

The exercise 2 in Assignment 9 outlines how to show, via method of integrating factor, that

$$m_{xy}(f,t) = i\,\gamma\,M_z^0\,e^{-ift}\int_0^t \alpha(s)e^{ifs}ds.$$

Given that we have assumed α is supported in $[0, \tau_p]$, this means that

$$m_{xy}(f, \tau_p) = 2\pi i\, \gamma\, M_z^0\, e^{-if\tau_p}\, \mathcal{F}_{ang}^{-1}(\alpha),$$

where $\mathcal{F}_{ang}^{-1}$ indicates the angular Fourier transform. Sometimes it is preferable to make an RF pulse symmetric in time, so we shift the envelop function α by $\tau_p/2$ to the left and denote it by $\alpha_{-\tau_p/2}$ defined by

$$\alpha_{-\tau_p/2}(t) = \alpha(t + \tau_p/2),$$

which has support in $[-\tau_p/2, \tau_p/2]$; now we work with the Fourier transform of the shifted function $\alpha_{-\tau_p/2}$. Once we have designed $\alpha_{-\tau_p/2}$, with the desired properties, we will also have α with those properties. To this end, we use the Fourier shift theorem for the angular transform to write

$$\begin{aligned} m_{xy}(f, \tau_p) &= 2\pi i\, e^{-if\tau_p}\, \gamma\, M_z^0\, \mathcal{F}_{ang}^{-1}(\alpha) \\ &= 2\pi i\, e^{-if\tau_p}\, \gamma\, M_z^0\, e^{if\tau_p/2}\, \mathcal{F}_{ang}^{-1}(\alpha_{-\tau_p/2}) \\ &= 2\pi i\, \gamma\, M_z^0\, e^{-if\tau_p/2} \mathcal{F}_{ang}^{-1}(\alpha_{-\tau_p/2}). \end{aligned}$$

The phase-shift $e^{-if\tau_p/2}$ can be undone by simply reversing $\boldsymbol{g}$ for a short time, say τ_{rev}, so that we can write

$$m_{xy}(f, \tau_p + \tau_{rev}) = 2\pi i\, \gamma\, M_z^0\, \mathcal{F}_{ang}^{-1}(\alpha_{-\tau_p/2}). \tag{2.9}$$

This can be done by *refocusing* (e.g., if initially the z-gradient G_z was a positive constant of duration τ_p (the duration of the RF pulse), then applying a negative gradient $-G_z$ for duration of $\tau_p/2$ right after the RF field is switched off will cancel this phase).

Setting the transverse magnetization from (2.9) equal to the desired slice selection profile in (2.8), and canceling common terms gives

$$2\pi\, \mathcal{F}_{ang}^{-1}(\alpha_{-\tau_p/2}) = \cos\theta\, \chi_{[-1,1]}\left(\frac{f}{\Delta\omega}\right) \equiv \cos\theta\, \chi_{[-\Delta\omega, \Delta\omega]}(f),$$

and taking the (angular) Fourier transform of both sides, we obtain

$$\alpha_{-\tau_p/2} = \mathcal{F}_{ang}\left(\chi_{[-\Delta\omega,\Delta\omega]}(f)\right)\, \cos\theta.$$

Hence, the small-flip-angle approximation to an ideal pulse is

$$\alpha(t) = \frac{\mathrm{sinc}(\pi\Delta\omega t)}{\Delta\omega}\, \cos\theta.$$

In practice, this sinc pulse must, of course, be truncated, which produces not an ideal slice selection but with artifacts at the edges, and there are various techniques how to deal with that (e.g., using a Hamming windowed sinc pulse), but we will not discuss it here.

To summarize, sinc pulse described above produces good approximation for excitation of up to 30° and acceptable excitations for profiles up to 90 degrees, although other methods that solve the problem numerically can produce significantly better results for flip angles much greater than 30 degrees. For larger flip angles, such as 180 degrees, used in more advanced imaging techniques like spin-echo imaging, the sinc approximation will not suffice, and a numerical solution is necessary.

We remark that, although slice selections which are horizontal are the most common, the method described above works to select slices with any orientation.

In practice, slice selection cannot be done with arbitrary accuracy. The slice thickness (for 2d imaging) in typical scanners is around 2-5mm [1], larger than the within-slice spatial resolution, which is usually about 1mm. This is one major disadvantage of attempting to perform volumetric imaging by successively imaging slices; the resulting 3D image will have lower resolution in one spatial direction than in the others.

[1] In advanced 3d imaging methods, the thickness can be 0.2-2mm