WEEK 8: GRADIENT FIELDS AND SIGNAL EQUATION

1. The Gradient Field and Interaction with the Background Field

Recall that the purpose served by the gradient field is to create small space-dependent modifications of the background field. Since $\|\boldsymbol{G}\|$ is typically around three orders of magnitude smaller than $\|\boldsymbol{B}_0\|$, geometric considerations make it clear that $\boldsymbol{G}$ will have no noticeable effects on the direction in which the overall field points. And a simple Taylor expansion of $\|\boldsymbol{B}\| = \|\boldsymbol{B} + \boldsymbol{G}\|$ (see exercises) shows that only the $z-$ component of $\boldsymbol{G}$ has a noticeable effect on the length of $\boldsymbol{B}$. Hence, we can ignore the components of $\boldsymbol{G}$ that are orthogonal to $\boldsymbol{B}_0$, and focus on the z-component. For this reason, $\boldsymbol{G}$ is sometimes written as

$$\boldsymbol{G} = (*, *, G_z),$$

where the asterisks indicate that we can ignore those components.

The gradient vector, although it is changed between stages in the excitation and measurement cycle, is generally constant throughout any given stage, and its z generally depends linearly on the location r, that is, there is a vector $\boldsymbol{g} \in \mathbb{R}^3$ such that

$$G_z(r) = \langle r, g \rangle = x\, g_x + y\, g_y + z\, g_z.$$

For example, if we want the gradient field to depend only on the z coordinate (as is the case in the slice-selection problem to follow), then we would let $\boldsymbol{g} = (0, 0, g_z)$, giving us $\boldsymbol{G}(r) = (*, *, z\, g_z)$. Modern MRI machines have specialized coils that can create the gradient field associated with any $\boldsymbol{g} \in \mathbb{R}^3$.

When both the main $\boldsymbol{B}_0$ field and the gradient field $\boldsymbol{G}$ are turned on, then we can express the z component of the magnetic field as $b(r)$, and then the solutions to the Bloch equation hold with $\omega(r) = \gamma b(r) = \gamma(b_0 + r \cdot \boldsymbol{g})$. We can also express the local resonance frequency $\omega(r)$ as

$$\omega(r) = \omega_0 + \Delta\omega(r),$$

where $\Delta\omega(r) = \gamma\, r \cdot \boldsymbol{g}$ is the local *frequency shift.* Then, in the rotating reference frame, we can solve for $\boldsymbol{m}_{xy}$ as

$$\boldsymbol{m}_{xy}(r, t) = e^{-t/T_2(r)}\, e^{-i\Delta\omega(r)t + i\phi}, \tag{1.1}$$

or, in the absence of relaxation terms, as

$$\boldsymbol{m}_{xy}(r, t) = e^{-i\Delta\omega(r)t + i\phi}.$$

It is worth noting that when $\phi = 0$ and $\boldsymbol{g}$ is constant then the resulting function of r is the kernel of the Fourier transform at the point $\boldsymbol{g} \in \mathbb{R}^3$.

2. Faraday's Law and the Fourier Transform Imaging

For the moment, let us take for granted that we can excite the magnetization, that is, that we can tip the bulk magnetization over to the xy-plane. In this section, the precise value of the z component of $\boldsymbol{M}(r, 0)$ is unimportant; we may assume without loss of generality that it is 0. We are interested only in times which are short relative to T_1 and T_2, so we may ignore the relaxation terms and assume that the bulk magnetization satisfies

$$\partial_t \boldsymbol{M}(r, t) = \boldsymbol{M}(r, t) \times \gamma\, \boldsymbol{B}(r, t). \tag{2.1}$$

Using the complex form for the transverse magnetization, suppose that $\boldsymbol{M}_{xy}(r, 0) = e^{i\phi}$. Recall, then, solution to equation (2.1) is

$$\boldsymbol{M}_{xy}(r, t) = |M_{xy}(r, 0)|\; e^{-t/T_2(r)}\, e^{-i\omega(r)t + i\phi(r)},$$

or, equivalently, in scalar form,

$$\begin{cases} M_x(r,t) &= |M_{xy}(r,0)|\; e^{-t/T_2(r)}\; \cos(-\omega(r)t + \phi(r)) \\ M_y(r,t) &= |M_{xy}(r,0)|\; e^{-t/T_2(r)}\; \sin(-\omega(r)t + \phi(r)). \end{cases} \tag{2.2}$$

2.1. **The Raw and Processed Signals.**

The transverse magnetization described above is detected using Faraday's Law. Suppose that we have placed a coil around an object—in the case of MRI, the RF coil is often used as the receiver coil—and that $\boldsymbol{B}_r(r)$ gives the magnetic field at point r that would hypothetically be produced by a unit direct current flowing in the coil. Then the *magnetic flux* through the coil given by $\boldsymbol{M}(r,t)$ is

$$\Phi(t) = \int_{\text{object}} \langle \boldsymbol{B}_R(r), \boldsymbol{M}(r,t) \rangle \, dr$$

Here, the R in the subscripts means 'receiver', emphasizing that the signal depends on the properties of the receiver coil. Faraday's law of induction states that the voltage $V(t)$ induced in the coil at time t is

$$V_r(t) = -\Phi'(t) = -\frac{d}{dt}\int_{\text{object}} \langle \boldsymbol{B}_R(r), \boldsymbol{M}(r,t) \rangle \; dr.$$

Now suppose that there are two coils, a coil detecting along the x-axis, with constant $\boldsymbol{B}_R^x(r) = (1,0,0)$ and coil detecting along the y-axis, with constant $\boldsymbol{B}_R^y(r) = (0,1,0)$. Then

$$V_x(t) = -\frac{d}{dt}\int_{\text{object}} M_x(r,t) dr = -\int_{\text{object}} \partial_t M_x(r,t) dr,$$

where we can bring the derivative inside the integral by Leibniz's rule because both M_x and $\partial_t M_x$ are continuous. Similarly,

$$V_y(t) = -\int_{\text{object}} \partial_t M_y(r,t) dr.$$

Next, we define a complex-valued *raw signal* or *induced voltage signal* V by setting

$$V(t) = V_x(t) + iV_y(t).$$

It follows from Equation (2.2) that

$$\begin{cases} \partial_t M_x(r,t) &= \omega(r)\, |M_{xy}(r,0)|\, e^{-t/T_2(r)} \sin(-\omega(r)t + \phi(r)) \\ & \quad -\dfrac{1}{T_2(r)}\, |M_{xy}(r,0)|\; e^{-t/T_2(r)}\; \cos(-\omega(r)t + \phi(r)) \\ \partial_t M_y(r,t) &= -\,\omega(r)\, |M_{xy}(r,0)|\, e^{-t/T_2(r)} \cos(-\omega(r)t) \\ & \quad -\dfrac{1}{T_2(r)}\, |M_{xy}(r,0)|\; e^{-t/T_2(r)}\; \sin(-\omega(r)t + \phi(r)). \end{cases} \tag{2.3}$$

In MRI, $\omega(r)$ is nearly always much larger than $1/T_2(r)$, so we can ignore the second term in the derivatives, yielding

$$\partial_t M_x(r,t) \approx \omega(r)\, |M_{xy}(r,0)|\; e^{-t/T_2(r)}\; \sin(-\omega(r)t + \phi(r))$$

and

$$\partial_t M_y(r,t) \approx -\omega(r)\, |M_{xy}(r,0)|\; e^{-t/T_2(r)}\; \cos(-\omega(r)t + \phi(r)),$$

which we can insert into the formula for $V(t)$ to get

$$V(t) \approx \int_{\text{object}} |M_{xy}(r,0)|\; e^{-t/T_2(r)} e^{-i(\omega(r)t + i\phi(r)} dr.$$

At this point, it is useful to replace $\omega(r)$ with

$$\omega_0 + \Delta\omega(r) = \omega_0 + \gamma r \cdot \boldsymbol{g},$$

where $r \cdot \boldsymbol{g}$ is the z component of the gradient field $\boldsymbol{G}$, that is, $G = (*, *, r \cdot \boldsymbol{g})$. Then we can pull the $e^{-i\omega_0 t}$ out of the integral, and the formula for the raw signal becomes

$$V(t) = e^{-i\omega_0 t} \int_{\text{object}} |M_{xy}(r, 0)|\, e^{-t/T_2(r)} e^{-i\gamma(r\cdot\boldsymbol{g})t + i\phi(r)} dr.$$

Thus far, we have assumed that the gradient vector g is constant during this process, but it will become useful later to allow it to vary, at least in a piecewise continuous manner. If $g(t)$ is a function of t, then we have $\Delta\omega(r, t) = \gamma r \cdot \boldsymbol{g}(t)$, and the term

$$-i\gamma(r \cdot \boldsymbol{g})t,$$

which is really an integral in t of the constant function $\Delta\omega(r) = \gamma r \cdot \boldsymbol{g}$, should be replaced by

$$\int_0^t \Delta\omega(r, \tau)\, d\tau = \int_0^t \gamma(r \cdot \boldsymbol{g}(t))\, d\tau = \gamma\, r \cdot \left(\int_0^t \boldsymbol{g}(t)\, d\tau \right),$$

where the third integral should be interpreted pointwise. The formula for the raw signal then becomes

$$V(t) = e^{-i\omega_0 t} \int_{\text{object}} |M_{xy}(r, 0)|\, e^{-t/T_2(r)} e^{-i\gamma r \cdot \int_0^t \boldsymbol{g}(t)\, d\tau + i\phi(r)} dr. \tag{2.4}$$

This is a high-frequency signal because the term in front of the integral changes at the Larmor frequency, but we can filter it to get a more useful low-frequency signal by multiplying it by $e^{-i\omega_0 t}$. We call the resulting processed signal $S(t)$. Then we have

$$S(t) = \int_{\text{object}} |M_{xy}(r, 0)|\, e^{-t/T_2(r)} e^{-i\,\gamma\, r \cdot \int_0^t \boldsymbol{g}(t)\, d\tau + i\phi(r)} dr. \tag{2.5}$$

To see the importance of this signal, let us make the usual assumption $\phi(r) = 0$ for all r. Further, suppose that g is constant, and that we are only looking at the signal over a period of time short enough that we can ignore the term containing T_2. Then the signal becomes

$$S(t) = \int_{\text{object}} |M_{xy}(r, 0)|\; e^{-ir\cdot(\gamma \boldsymbol{g} t)}\, dr,$$

which is precisely the (angular) Fourier transform of $|M_{xy}(r, 0)|$ at $k := \gamma \boldsymbol{g} t \in \mathbb{R}^3$, or equivalently, the unitary transform at $k = \dfrac{1}{2\pi}\gamma \boldsymbol{g}\, t$.

In the general case, as long as the time we spend reading the signal is small compared to T_2, then we will have

$$S(t) = \widehat{|M^0_{xy}|}(k), \quad \text{where} \quad k = \frac{\gamma}{2\pi} \int_0^t \boldsymbol{g}(t)\, d\tau. \tag{2.6}$$

The fact that $|M_{xy}(r, t)|$ has no clear meaning—or, rather, it turns out, several—is one of the great facts about MRI, because it allows for several means of creating contrast. For now, it will suffice to assume that excitation affects all magnetization equally, and sowe can interpret $|M_{xy}(r, t)|$ as the density $\rho(r)$ of hydrogen nuclei. It turns out that more involved excitation processes can turn $|M_{xy}(r, t)|$ into a function of not only $\rho(r)$, but also $T_1(r)$ and $T_2(r)$. This turns out to be great news for the usefulness of MRI, because experience has shown that nearly all human tissues differ in at least one of the three parameters ρ, T_1, and T_2.