WEEK 7: PHYSICS OF MRI

References for the topics discussed this week are primarily [1] and [3], and also [2].

1. Nuclear Spin and magnetic moment

The phenomenon of nuclear magnetic resonance (NMR) occurs in atoms with an odd number of protons or/and odd number of neutrons in a nucleus. Observations show that at least one unpaired proton or one unpaired neutron is needed to observe a **spin angular momentum**, denoted by $\mathbf{S}$, in an atomic nucleus. Spin numbers I are quantized to integers or half-integers and for several atomic nuclei are shown in Table 1. The spin angular momentum $\mathbf{S}$ is defined by

$$\mathbf{S} = \hbar \mathbf{I} = \frac{h}{2\pi}\mathbf{I},$$

where $h = 6.626 \times 10^{-34}$ kg $\cdot$ m^2/s is Planck's constant, $\hbar = h/2\pi$, and $\mathbf{I}$ is the spin quantum number (which has orientation $\pm$, we discuss that in details in the next section).

A nucleus with a nonzero spin ($I \neq 0$), and thus, a nonzero angular momentum, produces a current loop, and thus, creates a tiny magnetic field, called the *magnetic moment*, or *magnetic dipole*

$$\boldsymbol{\mu} = \gamma \mathbf{S} = \frac{\gamma}{2\pi} h\, \mathbf{I},$$

where γ is a gyromagnetic ratio and unique for different nucleus species (see Table 2).

Out of all atoms which exhibit NMR phenomenon the Hydrogen atom ^1H is the most abundant, it also has the largest gyromagnetic ratio, see Table 2. The coefficient γ is measured in MHz/T (million cycles per second per 1 Tesla). Here, T stands for Tesla, 1 Tesla $= 10^4$ Gauss. For a comparison, the magnetic field of Earth ranges $0.3 - 0.6$ G (it is stronger near the poles and weaker near the equator). From now on we will only consider ^1H as this is the nucleus currently used in MRI.

element	symbol	spin number I	atom configuration
hydrogen	^{1} H	1/2	1 proton, 0 neutrons
hydrogen	^{2} H	1	1 proton, 1 neutron
carbon	^{12} C	0	6 protons, 6 neutrons
carbon	^{13} C	1/2	6 protons, 7 neutrons
oxygen	^{16} O	0	8 protons, 8 neutrons
oxygen	^{17} O	5/2	8 protons, 9 neutrons
oxygen	^{18} O	0	8 protons, 10 neutrons
fluorine	^{19} F	1/2	9 protons, 10 neutrons
sodium	^{23} Na	3/2	11 protons, 12 neutrons
silicon	^{28} Si	0	14 protons, 14 neutrons
silicon	^{29} Si	1/2	14 protons, 15 neutrons
silicon	^{30} Si	0	14 protons, 16 neutrons
phosphorus	^{31} P	1/2	15 protons, 16 neutrons
chlorine	^{35} Cl	3/2	17 protons, 18 neutrons
chlorine	^{37} Cl	3/2	17 protons, 20 neutrons

Table 1. Spin numbers for some (stable) atomic nuclei.

element	symbol	$\gamma/2\pi$ (MHz/T)
hydrogen	^{1} H	42.575
carbon	^{13} C	10.7
fluorine	^{19} F	40.05
sodium	^{23} Na	11.26
phosphorus	^{31} P	17.235

Table 2. Gyromagnetic ratios for some atomic nuclei.

2. Interaction with Static Magnetic Field, Bulk Magnetization

Although there are both quantum-mechanical classical descriptions of the nuclear magnetic resonance (NMR) phenomenon at the atomic level, in MRI we are interested only in the macroscopic behavior of NMR. In the absence of an external magnetic field, the individual magnetic moments of the Hydrogen atoms in an object point in essentially random directions, so there is no resulting macroscopic magnetization. However, in the presence of an external magnetic field, the magnetic moments of the individual atoms will line up, with roughly half pointing up and roughly half pointing down (technically, they point up or down at $54°44'$, and then spin around the z-axis, but the details of this microscopic behavior are not important to MRI). This in an of itself would still lead to no macroscopic magnetization, if it were not for the fact that, at room temperature, slightly more atoms (on the order of one in a million, see calculations below) will point in the direction of the magnetic field rather than against it.

From now on we will assume that a (strong) static magnetic field $\boldsymbol{B}_0$ is oriented in z direction. The z direction is called *longitudinal* direction, and the xy - plane is called the *transversal* plane/direction.

When a sample is placed in a static magnetic field $\boldsymbol{B}_0$, magnetic dipoles orient themselves in the direction parallel or anti-parallel to this field. The parallel or anti-parallel direction of a magnetic dipole $\boldsymbol{\mu}$ corresponds to the orientation of its spin angular momentum $\mathbf{S}$ (and thus, $\mathbf{I}$) which in its turn is oriented in positive or negative direction of the magnetic field $\boldsymbol{B}_0$, thus, we have $\mathbf{I} = \pm 1/2$ (recall $I = 1/2$ in ${}^1H$). Denote by N_+ the number of nuclei in parallel direction with the magnetic field $\boldsymbol{B}_0$, and N_- the number of nuclei in anti-parallel direction with $\boldsymbol{B}_0$. From quantum mechanics it is known that in the magnetic field $\boldsymbol{B}_0 = (0, 0, B_0)$ a magnetic dipole $\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z)$ possesses a potential energy

$$E = -\boldsymbol{\mu} \cdot \boldsymbol{B}_0 = -\mu_z B_0 = -\frac{\gamma\, h}{2\pi}\, I B_0.$$

The two states of $I = \pm\frac{1}{2}$ lead to the two energy states

$$E_{1/2} = -\frac{\gamma h}{4\pi} B_0 \quad \text{and} \quad E_{-1/2} = \frac{\gamma h}{4\pi} B_0,$$

which are separated by

$$\Delta E = E_{-1/2} - E_{1/2} = \frac{\gamma h}{2\pi} B_0.$$

At the absolute zero ($T = 0$ Kelvin) all spins would orient themselves parallel to the static magnetic field $\boldsymbol{B}_0$. As temperature raises, some nuclei (in ${}^1H$ the nucleus consist only of one proton) will absorb photons (elementary particles which interact with matter by transferring energy), and thus, switch to a higher energy state $E_{-1/2}$ (since there only two states are possible). At a temperature T (in Kelvin) the ratio between N_+ and N_- is determined by Boltzmann distribution

$$\frac{N_-}{N_+} = e^{\Delta E / k_B T},$$

where $k_B = 1.381 \times 10^{-23}$ J/K is an absolute Boltzmann's constant measured in Joules per Kelvin (1 J = 1 kg $\cdot$ m^2/s^2). At an infinite temperature there would be equal amount of spins oriented parallel and anti-parallel, since $\frac{N_-}{N_+} = e^0 = 1$. At the body temperature $T = 37C = 99F = 310K$ and the magnetic field 1T,

$$\frac{N_-}{N_+} = .999993,$$

which means that the number of parallel oriented dipoles exceeds the number of dipoles with anti-parallel orientation only by 7 for each 10^6 dipoles oriented in parallel. This number looks tiny, but it is sufficient to measure the *bulk magnetization* $\mathbf{M}$ of the object, pointing in the same direction as the ambient field. Thus, we have $\mathbf{M} = \sum \boldsymbol{\mu}$.

Suppose that at a point $r = (x, y, z) \in \mathbb{R}^3$, the external magnetic field is $\boldsymbol{B}(r)$, and there are $N(r)$ hydrogen atoms in some small grid cube about r. Then the equilibrium bulk magnetization in that cube will take the form

$$\boldsymbol{M}^0(r) = \frac{\gamma^2 h^2}{16\pi^2 k_B}\, \frac{1}{T}\, N(r) \boldsymbol{B}(r).$$

Here, $N(r)$ is the total number of hydrogen atoms at the grid point, which, for comparison purposes, will be within a couple magnitudes of ten of Avogadro's number, 6.23×10^{26}, T is the temperature in Kelvin, which will be somewhere around 295 K at room temperature, and $\boldsymbol{B}(r)$ is the external magnetic field. Most of these constants are outside of our control. The only term determining $\boldsymbol{M}^0$ over which we do have some control is $\boldsymbol{B}$, and the fact that $\boldsymbol{M}^0$ is proportional to $\boldsymbol{B}$ is the main reason that MRI scanners use strong fields—typically ranging from 1.5 to 3 Tesla.

Typically, in order to model MRI in a continuous rather than discrete way, we instead use the related expression

$$\boldsymbol{M}^0(r) = \frac{C}{T}\, \rho(r)\, \boldsymbol{B}(r), \tag{2.1}$$

where C is constant, ρ is the hydrogen density near the point r, which we can think of as the average of $N(r)$ over progressively smaller neighborhoods around r. Our goal is to use measurements of the bulk magnetization $\boldsymbol{M}$ to reconstruct the density function ρ. There exists no way to directly measure $\boldsymbol{M}(r)$ at any point r in the scanner, therefore, in order to study the bulk magnetization in MRI, we first need to excite it, that is, get it moving in some manner which we can then detect. In order to do so, we need a model to describe the motion of a bulk magnetization not at equilibrium.

3. Precession and Relaxation processes

Initially we have bulk magnetization $\boldsymbol{M}$ oriented in the direction of $\boldsymbol{B}$. If $\boldsymbol{M}$ is dislocated from its initial position, then it starts to precess (or nutate) around the $\boldsymbol{B}$ direction The angular frequency of this precession is an important quantity and we discuss it next. In order to dislocate a magnetic moment $\boldsymbol{\mu}$ a torque $\boldsymbol{\tau}$ is applied to the dipole $\boldsymbol{\mu}$ in the presence of the external magnetic field $\boldsymbol{B}$ and is equal to their cross product

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}.$$

It is also equal to the rate of change of the (spin) angular moment of the dipole

$$\boldsymbol{\tau} = \frac{\partial \mathbf{S}}{\partial t}.$$

Combining the previous two expressions, we get

$$\frac{\partial \mathbf{S}}{\partial t} = \boldsymbol{\mu} \times \boldsymbol{B}.$$

Recalling that $\boldsymbol{\mu} = \gamma \mathbf{S}$, we obtain an equation for a single magnetic dipole

$$\frac{\partial \boldsymbol{\mu}}{\partial t} = \boldsymbol{\mu} \times \gamma \boldsymbol{B}.$$

Since the bulk magnetization is the sum of all magnetic moments, $\boldsymbol{M} = \sum \boldsymbol{\mu}$, we have

$$\frac{\partial \boldsymbol{M}}{\partial t} = \boldsymbol{M} \times \gamma \boldsymbol{B}. \tag{3.1}$$

In the Assignment # 7 we will show that $\boldsymbol{M}$ will precess about $\boldsymbol{B}$ with a frequency

$$\omega = \gamma B \quad \text{(rad/sec)},$$

or equivalently,

$$f = \frac{\gamma}{2\pi} B \text{ Hz} \quad (\equiv \text{cycles/sec}).$$

This frequency ω (or f – if measured in radians) is called the **Larmor** frequency and it is specific to a particular nuclei species (since it depends on γ). So we see that spins (and so magnetization) exhibit resonance which is understood through the precessional behavior at a well-defined frequency. To observe a chosen nuclear species in isolation from other nuclei, it suffices to tune on the "corresponding" Larmor frequency.

In a human body, hydrogen is present in water and in other biological macromolecules. It was observed that water has a distribution altered from normal in pathological tissues, and this allows to distinguish normal tissue from a pathology.

The dislocation of $\boldsymbol{M}$ from its equilibrium can be obtained by applying an additional magnetic field $\boldsymbol{B_1}$ in the transversal plane rotating at the Larmor frequency (otherwise, the two rotating fields would be out of phase and $\boldsymbol{M}$ would just slightly wobble). This magnetic field is the radio frequency field, sometimes also denoted by $\mathbf{B_{RF}}$. The reason it is called the radio frequency is because at the strength of $\boldsymbol{B} = 1$ Tesla, the Larmor frequency of the hydrogen atom ^1H is $f \approx 42.58$ million cycles per second or MHz which is in the range of the radio FM frequencies.

When an RF field $\boldsymbol{B_1}$ is applied, it is called an RF pulse, the angle between the equilibrium direction of $\boldsymbol{M}^0$ and $\boldsymbol{M}(t)$ increases as time goes on. If $\boldsymbol{B_1}$ is switched off as soon as the magnetization lies in xy-plane, it is called a 90^0 pulse. If this time is doubled (i.e., $\boldsymbol{B_1}$ is switched off when the magnetization changes the direction), then this pulse is called a 180^0 pulse.

Now suppose that we excite the magnetization so it precesses in the transversal plane (or we consider only the transversal component of the magnetization vector). If we place a coil at the top of the x-axis of an object, then the direct current will produce a magnetic field along the x-axis and an alternating current will make this magnetic field change along the x-axis. If we place another coil around y-axis, then the changing current will produce an changing magnetic field along the y-axis, and in combination with the first coil, the total magnetic field will be a sum of its x and y components, and thus, it will be possible to obtain a rotating at the Larmor frequency magnetic field $\boldsymbol{B_1}$.

The same coil which is used to generate the rotating magnetic field $\boldsymbol{B_1}$ can be used to detect the precessing magnetization. This detection is possible via the Faraday's Law of induction which we will discuss later. The resulting time signal, called Free Induction Decay (FID) signal, is the basic MR signal that is recorded.

Next we note that once the RF pulse is turned off, the magnetization vector will try to return to its equilibrium, and that will be done in two processes which are called the *relaxation* processes. The return of the longitudinal component M_z of the magnetization $\boldsymbol{M}$ to its original length M_0 is called the $T1$ relaxation, and the decay process in the transversal plane is called the $T2$ relaxation (since the transversal component M_{xy} will disappear in the equilibrium position). We study these processes next.

3.1. **T1 relaxation.** Consider only the longitudinal component M_z of the bulk magnetization vector $\boldsymbol{M}$. Suppose the original length at the equilibrium was M_0 and after a 90^0 RF pulse the magnetization completely lies in the xy plane, thus, lacking the z component. This behavior of $M_z = M_z(t)$ is described by the following initial value problem:

$$\frac{dM_z}{dt} = \frac{M_0 - M_z}{T_1}, \tag{3.2}$$

$$M_z(0) = 0. \tag{3.3}$$

Here, T_1 is the constant of proportionality. Solving the equation (3.2), for example, by separation of variables, we obtain

$$M_z - M_0 = (M_z(0) - M_0)\, e^{-t/T_1},$$

and thus,

$$M_z(t) = M_0 \left(1 - e^{-t/T_1}\right).$$

Observe that at the time $t = T_1$,

$$M_z(T_1) = M_0 \left(1 - \frac{1}{e}\right) \approx .63 M_0,$$

which means that T_1 is the time required for the z-component of $\boldsymbol{M}$ to return to 63% of it original magnitude. The T_1 constant is referred to the spin-lattice relaxation time constant. The name 'spin-lattice' indicates that in this relaxation process there is an energy exchange between a spin and a lattice around it. Because of the interaction with the lattice, and thus, with the magnetic field which orients it, T_1 depends on the external magnetic field: if $\boldsymbol{B}_0$ increases, then T_1 increases too.

3.2. **T2 relaxation.** Now we study the transversal component M_{xy} of the bulk magnetization $\boldsymbol{M}$. This component decays with time, and thus, can be described by the following initial value problem:

$$\frac{dM_{xy}}{dt} = -\frac{M_{xy}}{T_2}, \tag{3.4}$$

$$M_{xy}(0) = 0. \tag{3.5}$$

For now, T_2 is merely a constant of proportionality. Solving the equation (3.4), we obtain

$$M_{xy}(t) = M_0\, e^{-t/T_2}.$$

At time $t = T_2$, the transversal component is

$$M_{xy}(T_2) = \frac{M_0}{e} \approx .37 M_0,$$

which means that T_2 is the time required for the transversal component of $\boldsymbol{M}$ to decay to 37% of its initial value. The constant T_2 is also called 'spin-spin' relaxation time constant indicating that the energy exchange is taking place only between spins and this decays happens because of dephasing of the transversal component in the bulk magnetization M_{xy}. Thus, T_2 is basically independent of the magnetic field $\boldsymbol{B}_0$ strength.

In solids the T_2 decay is very rapid as well as in some biological tissues where spins are sounded to the lumbering macromolecules. Those spins which correspond to water have much slower T_2 decay, this allows to distinguish various tissues from each other.

Finally we have arrived to the dynamics of the nuclear magnetization which is phenomenologically described by the Bloch equation. We discuss it next.

4. The Bloch Equation and Free Precession

The Bloch equation gives a classical description of the non-equilibrium behavior of a bulk magnetization in the presence of an external magnetic field. In the presence of a (possibly changing) magnetic field $\boldsymbol{B}(r,t)$, the bulk magnetization $\boldsymbol{M}(r,t)$ behaves as follows

$$\frac{\partial \boldsymbol{M}(r,t)}{\partial t} = \boldsymbol{M}(r,t) \times \gamma \boldsymbol{B}(r,t) - \frac{\boldsymbol{M}_{\mathbf{xy}}(r,t)}{T_2(r)} - \frac{\boldsymbol{M}_{\boldsymbol{z}} - \boldsymbol{M}^0(r,t)}{T_1(r)}, \tag{4.1}$$

or writing out the second and third terms explicitly,

$$\frac{\partial \boldsymbol{M}(r,t)}{\partial t} = \boldsymbol{M}(r,t) \times \gamma \boldsymbol{B}(r,t) - \frac{M_x(r,t)\,\boldsymbol{\imath} + M_y(r,t)\,\boldsymbol{\jmath}}{T_2(r)} - \frac{M_z(r,t)\,\boldsymbol{k} - \boldsymbol{M}^0(r,t)}{T_1(r)}, \tag{4.2}$$

where $\boldsymbol{M}^0(r,t)$ is the equilibrium magnetization defined by the equation (2.1) at time t, and $\boldsymbol{M}_{\boldsymbol{z}}$ is the *longitudinal magnetization* of $\boldsymbol{M}$, and $\boldsymbol{M}_{\mathbf{xy}}$ is called the *transverse magnetization*.

The first term, $\boldsymbol{M} \times \gamma \boldsymbol{B}(r,t)$ comes from the direct interaction of the spins with the magnetic field, see equation (3.1). The other two terms, the transverse and longitudinal *relaxation terms*, come from averaging tiny interactions between microscopic magnetic fields within the object which drive the system back to equilibrium, these terms are those showing in the equations (3.2) and (3.4). Recall that the T_1 term forces the longitudinal component of $\boldsymbol{M}$ back into equilibrium, and the T_2 term forces the transverse component of $\boldsymbol{M}$ to zero as $t \to \infty$. We added arguments to the T_1 and T_2 terms to emphasize that in practice they are functions of location; the fact that these constants differ between different bodily tissues, due to the different chemical properties that surround the hydrogen nuclei, is an important source of contrast in MRI.

The solution to the equation (4.1) when the magnetic field is constant in time and points in the z direction is central to the theory of MRI. We derive it in the Assignment # 7 and recall it here: suppose $\boldsymbol{B}(r,t) = (0,0,b(r))$ is constant (in time) for each r. Then, given an initial magnetization $\boldsymbol{M}(r,0)$ at location r, the solution to the Bloch equation is [1]

$$\begin{cases} \boldsymbol{M}_{xy}(r,t) &= e^{-t/T_2(r)} \begin{pmatrix} \cos(\omega(r)t) & \sin(\omega(r)t) \\ -\sin(\omega(r)t) & \cos(\omega(r)t) \end{pmatrix} \boldsymbol{M}_{xy}(r,0), \\ \\ M_z(r,t) &= (1 - e^{-t/T_1(r)})\, M_z^0(r) - e^{-t/T_1(r)}\, M_z(r,0), \end{cases} \tag{4.3}$$

where $\omega(r) = \gamma b(r)$ is the *Larmor frequency* at the point r, recall it is the rate at which the bulk magnetization precesses around the vertical axis. The equation (4.3) is called the *free precession* equation. Its solution is a spiral around the z axis which converges to $\boldsymbol{M}^0(r)$ as $t \to \infty$. The constants T_1 and T_2 give the rates of relaxation of the longitudinal and transverse components. Although the precise values of T_1 and T_2 vary for different tissues, T_1 is always much larger than T_2, so that the transverse magnetization decays much faster than the longitudinal magnetization recovers.

When the external field is as strong as that encountered in MRI, first term acts much faster than the other two. Hence, in cases when we are only interested in the behavior of the magnetization over times relatively short compared to T_1 and T_2, we can ignore the relaxation terms from the Bloch equation, and use the following approximation:

$$\frac{\partial \boldsymbol{M}(r,t)}{\partial t} = \gamma \boldsymbol{M}(r,t) \times \boldsymbol{B}(r,t) \tag{4.4}$$

The solution to the equation (4.4) under the same conditions as before is

$$\boldsymbol{M}(r,t) = \begin{pmatrix} \cos(\omega(r)t) & \sin(\omega(r)t) & 0 \\ -\sin(\omega(r)t) & \cos(\omega(r)t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{M}(r,0). \tag{4.5}$$

The matrix above is referred to the *solution operator* of the Bloch equation without relaxation terms.

We close this section by introducing a convention which can make it easier to solve some of the differential equations associated with MRI, and also becomes extremely useful in signal detection, since it highlights the connection between MRI and the Fourier transform. In MRI, our primary interest is usually in $\boldsymbol{M}_{xy}$. Since scalars tend to be easier to work with than vectors, it is convenient to be able to treat $\boldsymbol{M}_{xy}$ as a scalar rather than a vector. So, instead of viewing $\boldsymbol{M}_{xy}$ as a vector in $\mathbb{R}^2$, we view it as an element of the complex plain, i.e.,

$$M_{xy} = M_x + iM_y$$

(note that we do not boldface M_{xy} when treating it as a complex number).

Suppose that $\boldsymbol{M}(r,0)$ lies in the xy plane, and it makes an angle ϕ with the positive x axis, sometimes called the *phase shift.* Then (the transverse part of) the equation (4.3) is

$$M_{xy}(r,t) = |M_{xy}(r,0)|\, e^{-t/T_2(r)} e^{-i\omega(r)+i\phi},$$

and (4.5) becomes

$$M_{xy}(r,t) = |M_{xy}(r,0)|\, e^{-i\omega(r)+i\phi}.$$

5. The $\boldsymbol{B}_0$ Field and the Rotating Frame of Reference

Let $\boldsymbol{B}_0 = (0,0,b_0)$, and let $\omega_0 = \gamma b_0$. Then $\boldsymbol{B} = \boldsymbol{B}_0$ is uniform, and the solution operator to the Bloch equation (4.4) in the absence of relaxation terms, is

$$\boldsymbol{M}(r,t) = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{M}(r,0). \tag{5.1}$$

This behavior, and the similar behavior accounting for relaxation terms, is central to MRI. But the various stages in MRI require manipulating the bulk magnetization using the other fields, and the Bloch equation is difficult to solve

[1] The rotation here is in the opposite direction of that specified in equation (14.9) of Epstein's test. This is an error in the Epstein text, although in reality it does not matter much, since it just amounts to a reversal of one of the axes

when these fields interact with the stronger field. This problem is solved by introducing a *rotating frame of reference* which neutralizes the effects of the strong field.

Let $\boldsymbol{U}(t)$ be the solution operator in equation (5.1), that is, let $\boldsymbol{M}(r,t) = \boldsymbol{U}(t)\boldsymbol{M}(r,0)$. Our goal is to cancel the effect of $\boldsymbol{U}(t)$, so we define the rotating reference operator as

$$\boldsymbol{W}(t) = [\boldsymbol{U}(t)]^{-1} = \begin{pmatrix} \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ \sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define axes x', y', z' by setting

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \boldsymbol{W}(t) \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

so that the magnetic moment vector in the rotating frame, which we denote by $\boldsymbol{m}$, by

$$\boldsymbol{m}(t) = \boldsymbol{W}(t)\boldsymbol{M}(t).$$

It is easy to see directly from the definitions that under free precession without relaxation terms,

$$\boldsymbol{m}(t) = \boldsymbol{W}(t)\boldsymbol{M}(t) = \boldsymbol{W}(t)\boldsymbol{U}(t)\boldsymbol{M}_0 = \boldsymbol{I}\boldsymbol{M}(0) = \boldsymbol{M}(0),$$

so that free precession in the rotating reference frame just means holding a constant position.

It can be shown via the product rule that if $\boldsymbol{M}$ satisfies the Bloch equation, then for $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$, the magnetization in the rotating frame of reference $\boldsymbol{m} = (m_x, m_y, m_z)$ with $\boldsymbol{m_{xy}} = m_x\boldsymbol{\imath} + m_y\boldsymbol{\jmath}$ behaves according to

$$\frac{\partial \boldsymbol{m}(r,t)}{\partial t} = \boldsymbol{m}(r,t) \times \gamma \boldsymbol{B}_{\text{eff}}(r,t) - \frac{\boldsymbol{m_{xy}}(r,t)}{T_2(r)} - \frac{(m_z(r,t) - M_z^0(r))\boldsymbol{k}}{T_1(r)},$$

where $\boldsymbol{B}_{\text{eff}} = \boldsymbol{W}(t)\boldsymbol{B} - (0, 0, \frac{\omega_0}{\gamma})$. This equation considerably simplifies the solutions to problems related to exciting the magnetic spins (that is, rotating them down to the xy plane).

6. Magnetic fields in MRI

In general, the external magnetic field in an MRI scanner takes the form

$$\boldsymbol{B}(r,t) = \boldsymbol{B}_0 + \boldsymbol{B}_{RF}(t) + \boldsymbol{G}(r,t),$$

where $\boldsymbol{B}_0 = (0, 0, b_0)$ is a time and space independent *background field*, $\boldsymbol{B}_{RF}(t)$ is a time-dependent *radio frequency field* which is either zero or is perpendicular to $\boldsymbol{B}_0$ and rapidly oscillating (with Larmor frequency), and $\boldsymbol{G}$ is a space-dependent *gradient field* which is usually taken to be "piecewise constant", that is, changing from time to time but constant in the short run.

The $\boldsymbol{B}_0$ field is far stronger than the other two fields, generally in the 1.5-3 Tesla range. There are two reasons why the background field is so strong. One is that the macroscopic bulk magnetization it creates, according to the relation in the equation (2.1), is proportional to its strength. The other reason is that the signal eventually measured in MRI comes not directly from the magnetization, but, via Faraday's law, its rate of change. And, as the equations (4.3) and (4.5) demonstrate, the rate of change is also proportional to the strength of the external magnetic field. For these reasons, the final signal measured in MRI is proportional not to $\boldsymbol{B}_0$, itself, but rather to its *square*. Hence, the strength of the background field is extremely important in determining how much resolution can be achieved in MRI.

The purpose of the radio frequency field, or RF field, is to excite the bulk magnetization. If we simply place a patient in an MRI scanner with the background field turned on, then the hydrogen nuclei in the patient will give a macroscopic magnetization according to equation (2.1), but we need the magnetization to be moving in order to detect them and use them for imaging. The RF field, which oscillates at the same speed as the bulk magnetization do in equation (4.5), moves the bulk magnetization away from its equilibrium state—often 90° to the xy-plane — so that the solution in equation (4.3) exhibits nontrivial motion.

The purpose of the gradient field is to create small space-dependent alterations in the strength of the background field that allow us to collect data that differentiate between different locations within the patient. We will discuss this more next week.

References

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[2] Paul C. Lauterbur and Zhi-Pei Liang. *Principles of Magnetic Resonance Imaging: A Signal Processing Perspective.* IEEE Press, New York, 2000.

[3] Dwight G. Nishimura. *Principles of Magnetic Resonance Imaging.* 1996. Lecture Notes from a Stanford University Course.