WEEK 15: APPLICATION OF PROBABILITY TO MEDICAL IMAGING

1. Probabilistic Approach to Beer's Law

Recall that Beer's Law is the main assumption made in tomography about the attenuation of intensity. To show that it is an adequate rule we will derive this law using the probabilistic approach. The rough outline of this approach is in the textbook by Epstein, but here we will give detailed explanations, some of them are listed as exercises which you might want to attempt thinking on your own first and then read the explanation.[1]

Consider N photons traveling through a material along an interval of $[a, b]$, which is contained in a line l. Let χ_N be a random variable that equals the number of photons that are emitted from the object. The attenuation coefficient μ is a nonnegative function which is defined along the line l, i.e., $\mu = \mu(s)$. Let Δs be a very small distance on our line, which eventually will be assumed to be infinitesimal. We assume that photons are independent of each other. For each photon, the following are true:

- A single particle that is incident upon the material at point s has a probability $\mu(s)\Delta s$ of being absorbed and probability $1 - \mu(s)\Delta s$ of being emitted.
- Each particle is independent of each other particle.
- Disjoint subintervals of $[a, b]$ are independent.

The first step is to divide $[a, b]$ into m (equal) subintervals, denote the kth subinterval by J_k.

Exercise 1.1. Write explicitly J_k.

Solution. If m is the number of intervals and the total length is $b - a$, then each interval J_k, $k = 1, 2, ..., m$, should have a length of $\frac{b-a}{m}$. Also the interval where $k = 1$ should start at a, where as the interval where $k = m$ should end at b. These should lead to the following interval:

$$J_k = \left[a + \frac{(k-1)(b-a)}{m}, a + \frac{k(b-a)}{m}\right].$$

□

The particle under consideration, in order to get to b, must pass through every subinterval J_k.

Exercise 1.2. Derive the probability[2] $p_{k,m}$ of a particle that passes through J_k (recall the probability $(1 - \mu(s)\Delta s)$). Notice that $\mu(s)$ only takes in the point, where we are dealing with intervals. Use the back end of the interval to plug into the $\mu(s)$.

Solution. Since

$$J_k = \left[a + \frac{(k-1)(b-a)}{m}, a + \frac{k(b-a)}{m}\right] \quad \text{and} \quad \Delta s = \frac{b-a}{m},$$

we obtain (with $s = a + \frac{k(b-a)}{m}$)

$$p_{k,m} = 1 - \mu\left(a + \frac{k(b-a)}{m}\right)\left(\frac{b-a}{m}\right).$$

□

[1] The idea for this worksheet was suggested and partially written by the undergraduate David Weis.

[2] Here, the subscript indicates that there are m subintervals.

As the probability of the particle passing through all these intervals is just the product of each of the $p_{k,m}$, we have the probability $p_{ab,m}$ of the particle passing through the entire interval as being

$$p_{ab,m} \approx \prod_{k=1}^{m} p_{k,m}.$$

It is an approximation because of the fact that we must let m tend to infinity for this statement to be completely valid. Now suppose that μ is a constant function: $\mu = \mu_0$.

Exercise 1.3. Show that $\lim_{m\to\infty} \prod_{k=1}^{m} p_{k,m} = e^{-\mu_0(b-a)}$ if μ is the constant function μ_0.

Solution. If μ is the constant function μ_0, then

$$p_{k,m} = 1 - \mu_0 \frac{(b-a)}{m}.$$

This does not depend on k, so that

$$\prod_{k=1}^{m} p_{k,m} = \left(1 - \mu_0 \frac{(b-a)}{m}\right)^m.$$

Now taking the limit,

$$\begin{aligned}\lim_{m\to\infty} \prod_{k=1}^{m} p_{k,m} &= \lim_{m\to\infty} \left(1 - \frac{\mu_0(b-a)}{m}\right)^m \\ &= e^{-\mu_0(b-a)}.\end{aligned}$$

This is an elementary result from calculus $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$. □

This shows that a single particle which enters a material at a has probability $e^{-\mu_0(b-a)}$ of being emitted at b.

Exercise 1.4. Find an expression for the probability that k out of N photons emerge at point b. (Hint: this is simply a binomial distribution).

Solution. As the hint indicates, this is a binomial distribution with probability of success $e^{-\mu_0(b-a)}$, number of trials N, and number of successes k. Then

$$P(k; N) = \left(\begin{array}{c} N \\ k \end{array}\right) e^{-k\mu_0(b-a)}(1 - e^{-\mu_0(b-a)})^{N-k}.$$

□

Exercise 1.5. From this, prove that the signal-to-noise ratio for this experiment is

$$SNR = \sqrt{N\left(\frac{e^{-\mu_0(b-a)}}{1 - e^{-\mu_0(b-a)}}\right)}.$$

(Recall that if a distribution has mean μ and standard deviation σ, it's SNR $= \frac{\mu}{\sigma}$.

Proof. It is good that we have determined that this experiment follows a binomial distribution since there are well known facts about a binomial distribution which we are able to use. For example, if a binomial distribution has parameters n and p, where n is the number of trials and p is the probability of success, then it is well known that the mean of this distribution is np and the variance is $np(1-p)$. So if we denote the mean $E(\chi)$ and variance σ^2, then

$$E(\chi) = Ne^{-\mu_0(b-a)} \quad \text{and} \quad \sigma^2 = Ne^{-\mu_0(b-a)}(1 - e^{-\mu_0(b-a)}).$$

Then the SNR is given as

$$\begin{aligned}
\text{SNR} &= \frac{Ne^{-\mu_0(b-a)}}{\sqrt{Ne^{-\mu_0(b-a)}(1 - e^{-\mu_0(b-a)})}} \\
&= \frac{N}{\sqrt{N}} \frac{e^{-\mu_0(b-a)}}{\sqrt{e^{-\mu_0(b-a)}}} \frac{1}{\sqrt{1 - e^{-\mu_0(b-a)}}} \\
&= \sqrt{N}\sqrt{e^{-\mu_0(b-a)}} \frac{1}{\sqrt{1 - e^{-\mu_0(b-a)}}} \\
&= \sqrt{N \frac{e^{-\mu_0(b-a)}}{1 - e^{-\mu_0(b-a)}}}.
\end{aligned}$$

□

Remark 1.6. This is a useful result. First, as N increases (which means more particles are being shot at the material), it is easy to see that the SNR increases, and so we interpret this as saying that the quality of the measurements increases. Also the higher μ_0 is (which means more particles are being absorbed), the smaller the SNR is, and so the less quality our measurements have. Harder material tends to absorb more particles, and so this result says that those measurements will have less quality then those dealing with softer materials.

Now we will proceed by lifting the assumption that μ is a constant.

Exercise 1.7. For the equation $p_{ab,m} \approx \prod_{k=1}^{m} p_{k,m}$, write out the right hand side explicitly.

Solution. Since

$$p_{k,m} = 1 - \mu\left(a + k\frac{(b-a)}{m}\right)\left(\frac{b-a}{m}\right),$$

we have

$$\begin{aligned}
p_{ab,m} &\approx \prod_{k=1}^{m}\left(1 - \mu\left(a + k\frac{(b-a)}{m}\right)\left(\frac{b-a}{m}\right)\right) \\
\implies \ln p_{ab,m} &\approx \ln \prod_{k=1}^{m}\left(1 - \mu\left(a + k\frac{(b-a)}{m}\right)\left(\frac{b-a}{m}\right)\right) \\
\implies \ln p_{ab,m} &\approx \sum_{k=1}^{m} \ln\left(1 - \mu\left(a + \frac{k(b-a)}{m}\right)\left(\frac{b-a}{m}\right)\right),
\end{aligned}$$

where the last line is an extension of the property of the natural logarithm that $\ln(xy) = \ln(x) + \ln(y)$. □

To continue, recall that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \tag{1.1}$$

for $|x| < 1$. This is the Taylor expansion of $\ln(1-x)$. The reason we want to observe this is because in order to make our equation exactly equal, we need to take the limit as $m \to \infty$, as we did in the case when we took μ as a constant. When using the Taylor expansion, we will see that most of the Taylor expansion decays like or faster than $\frac{1}{m}$, and so we can remove it from our equation as it will approach zero.

Exercise 1.8. Write out the first three terms of the Taylor expansion of

$$\ln\left(1 - \mu\left(a + \frac{k(b-a)}{m}\right)\frac{b-a}{m}\right).$$

What pattern emerges that will allow you to consider only the first term of each of these expressions for a given k?

Solution. Substituting $\mu\left(a + k\frac{(b-a)}{m}\right)\frac{b-a}{m}$ for x in the equation (1.1), we obtain that the first three terms are

$$-\left(\mu\left(a+k\frac{(b-a)}{m}\right)\frac{b-a}{m} + \frac{1}{2}\mu\left(a+k\frac{(b-a)}{m}\right)^2\left(\frac{b-a}{m}\right)^2\right. \tag{1.2}$$
$$\left. +\frac{1}{3}\mu\left(a+k\frac{(b-a)}{m}\right)^3\left(\frac{b-a}{m}\right)^3 + \ldots\right).$$

If we let m tend to infinity, all terms in (1.2) will decay to zero but with different rates. We will denote the decay as $O(m^{-l})$, meaning that the terms decay as m^{-l} when m tends to infinity. Note that the first term in (1.2) behaves as $O(m^{-1})$, the second one as $O(m^{-2})$, the third one as $O(m^{-3})$ and so on. Since the first term decays the slowest, we will keep only that one and write $O(m^{-2})$ for all other ones in the equation (1.2). Thus, considering the equation (1.2) for each k and using the O notation, we obtain

$$-\mu\left(a + \frac{k(b-a)}{m}\right)\frac{b-a}{m} + O(m^{-2}).$$

□

Exercise 1.9. Now rewrite the sum using only the first term of the Taylor series for each k, denoting all other terms as $O(m^{-2})$. Take the limit as $m \to \infty$. This will make[3] $p_\mu = \lim_{m\to\infty}\prod_{k=1}^{m} p_{k,m}$, so it is no longer an approximation but the exact equality.

Solution. We have

$$\ln p_{ab,m} \approx \sum_{k=1}^{m}\left(-\mu\left(a + \frac{k(b-a)}{m}\right)\frac{b-a}{m} + O(m^{-2})\right).$$

Then

$$\begin{aligned}\lim_{m\to\infty}\ln p_{ab,m} &= \lim_{m\to\infty}\sum_{k=1}^{m}\left(-\mu\left(a + \frac{k(b-a)}{m}\right)\frac{b-a}{m}\right) + \lim_{m\to\infty}\sum_{k=1}^{m} O(m^{-2})\\ &= \lim_{m\to\infty}\sum_{k=1}^{m}\left(-\mu\left(a + \frac{k(b-a)}{m}\right)\frac{b-a}{m}\right) + \lim_{m\to\infty} O(m^{-1}),\end{aligned}$$

[3]The notation p_μ is used to be consistent with the textbook, it is still the probability that a particle incident at a emerges at b, but since we took the limit as $m \to \infty$ there is no division of the interval to speak of anymore.

and thus,

$$\ln p_\mu = \lim_{m\to\infty} \sum_{k=1}^{m} \left(-\mu \left(a + \frac{k(b-a)}{m} \right) \frac{b-a}{m} \right). \tag{1.3}$$

Now the right hand side of (1.3) is the definition of a Riemann integral of $-\mu$ over $[a, b]$, and hence, can be written as

$$\ln(p_\mu) = -\int_a^b \mu(s)ds. \tag{1.4}$$

□

Exercise 1.10. Next, find p_μ from (1.4).

Solution. By (1.4) we have

$$\exp(\ln p_\mu) = \exp\left[-\int_a^b \mu(s)ds \right] \implies p_\mu = \exp\left[-\int_a^b \mu(s)ds \right].$$

□

We now have found the probability that a particle incident at a emerges at b when μ is not a constant.

Exercise 1.11. As we did in the case when μ is a constant, find the probability that k out of N incident photons emerge from the material.

Solution. As before this is a binomial distribution with N trials and probability of success p_μ:

$$P(k; N) = \binom{N}{k} (p_\mu)^k (1 - p_\mu)^{N-k}.$$

□

Exercise 1.12. Finally, find the expected number of particles to emerge. (This will give you Beer's Law).

Solution. The expected value of this experiment is

$$E[\chi_N] = Np_\mu = N \exp\left[-\int_a^b \mu(s)ds \right],$$

which is, in fact, Beer's law. □

It is pretty amazing to see this physics problem has an explicit probabilistic proof.

To finish this discussion, we will mention the variance of the Bernoulli random variable χ_N and its SNR:

$$var(\chi_N) = p_\mu(1 - p_\mu)N,$$

and

$$SNR(\chi_N) = \sqrt{N\left(\frac{p_\mu}{1 - p_\mu}\right)},$$

the last expression is calculated similarly to the one in Exercise 1.5.