494/598 Fall 2008

ASSIGNMENT 9: EXCITATION

NONSELECTIVE EXCITATION

First, we work out how to design a nonselective RF pulse for a given flip angle. We assume that the radio frequency field $\boldsymbol{B}_{RF}(r,t)$ takes the form

$$\boldsymbol{B}(r) = (\alpha(t)\cos\omega_0 t, -\alpha(t)\sin\omega_0 t, 0)^t.$$

In the next set of exercises we will show that the solution in the rotating frame in this situation is

$$\boldsymbol{m}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & \sin\theta(t) \\ 0 & -\sin\theta(t) & \cos\theta(t) \end{pmatrix} \boldsymbol{m}(0), \tag{1}$$

where the flip angle $\theta(t)$ is defined as

$$\theta(t) = \gamma \int_0^t \alpha(\tau)\, d\tau. \tag{2}$$

In the above equation (1) we suppressed the dependence on the location r, but it should be understood that $\boldsymbol{m} = \boldsymbol{m}(r,t)$; however, since this is a non-selective excitation, there is no distinguishing between different locations.

Although the solution could be found multiple ways, the easiest is to note that one component (the x component m_1) has a trivial solution, and the solution for the other two can be found by treating the remaining two components as a single complex number, as we did for the third method of solving the Bloch equation without relaxation terms in the previous assignment.

Problem 1a. Derive $\boldsymbol{B}_{\text{eff}}$ in the case when

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_{RF}(r) = (0,0,b_0)^t + (\alpha(t)\cos\omega_0 t, -\alpha(t)\sin\omega_0 t, 0)^t.$$

Recall that $\boldsymbol{B}_{\text{eff}} = \boldsymbol{W}(t)\,\boldsymbol{B} - (0,0,b_0)^t$ and $\boldsymbol{W}(t) = \begin{pmatrix} \cos\omega_0 t & -\sin\omega_0 t & 0 \\ \sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Next, we will solve the Bloch equation which we derived in the previous assignment but without relaxation terms, since as we observed in class on Wednesday the duration of an RF pulse is much less then $T1$ or $T2$ times. Recall that the equation is

$$\frac{\partial \boldsymbol{m}(r,t)}{\partial t} = \boldsymbol{m}(r,t) \times \gamma \boldsymbol{B}_{\text{eff}}(r,t), \tag{3}$$

where $\boldsymbol{m}(r,t) = (m_1(r,t), m_2(r,t), m_3(r,t))^t$.

Problem 1b. Show that the equation (3) is equivalent to the system

$$\begin{cases} \dfrac{\partial m_1}{\partial t} &= 0, \\ \dfrac{\partial m_2}{\partial t} &= \gamma\,\alpha(t)\,m_3, \\ \dfrac{\partial m_3}{\partial t} &= -\gamma\,\alpha(t)\,m_2. \end{cases} \tag{4}$$

The first equation in (4) has a constant solution (in time), and thus (suppressing the dependence on r),

$$m_1(t) = m_1(0). \tag{5}$$

Problem 1c. Consider the remaining two equations in (4), represent the $\mathbb{R}^2$-valued vector of these components $(m_2, m_3)^t$ as an element of $\mathbb{C}$ ($m := m_2 + i m_3$). Now, show that the derivative of this complex-valued function m can be expressed as a complex-valued (non-constant) coefficient times the original function.

Next, use separation of variables to solve the differential equation you just obtained for m. You should obtain $m(t) = e^{-i\,\theta(t)}\, m(0)$, where θ is as in (2), and combined with (5) and converted back into $\mathbb{R}^3$ form, should match the equation (1).

Slice Selection or Selective Excitation

Recall that on Wednesday lecture, we derived a solution to the 'small-flip-angle' problem by assuming that m_3 is almost unchanged from the equilibrium value M_z^0 and treating the $\mathbb{R}^2$-valued function $\boldsymbol{m}_{xy}$ as a complex-valued function $m_{xy} = m_1 + i\, m_2$. In this case, a differential equation describing the behavior of m_{xy} is

$$\frac{\partial m_{xy}(\omega, t)}{\partial t} = -i\,\omega\, m_{xy}(\omega, t) + i\gamma\, \alpha(t)\, M_z^0. \tag{6}$$

We are now ready to use this equation (6) to derive the Fourier relationship between the RF pulse and the slice selection profile. In class I mentioned that this problem can be solved by an integrating factor method from ODE, also by the Laplace transform and by the Fourier transform.

Problem 2. In this exercise, we will solve the equation (6) using an integrating factor method. Assume that $\alpha(t)$ is supported on an interval $[0, \tau_p]$ and m'_{xy} is understood as the time derivative of m_{xy}. We rearrange (6) to get

$$m'_{xy}(\omega, t) + i\,\omega\, m_{xy}(\omega, t) = i\gamma\, \alpha(t)\, M_z^0. \tag{7}$$

Our initial condition is $\boldsymbol{m}(\omega, 0) = \boldsymbol{M}^0$, i.e.,

$$m_{xy}(\omega, 0) = 0.$$

Find a function $\mu(t)$ such that

$$\frac{d}{dt}[\mu(t)\, m_{xy}(\omega, t)] = [m'_{xy}(\omega, t) + i\,\omega\, m_{xy}(\omega, t)]\, \mu(t). \tag{8}$$

(*Use the product rule on the left hand side, and obtain a differential equation for μ, then solve it by separation of variables. The result should be the familiar kernel of an angular Fourier transform: $e^{i\,\omega\, t}$.*)

Next, make use of the integrating factor. Multiply both sides of the equation (7) by $\mu(t)$, and write the left hand side as a full derivative $\frac{d}{dt}[e^{i\,\omega\, t}\, m_{xy}(\omega, t)]$. Then integrate in time from 0 to t to obtain

$$e^{i\,\omega\, t}\, m_{xy}(\omega, t) = i\,\gamma\, M_z^0 \int_0^t \alpha(s)\, e^{i\,\omega\, s}\, ds + C.$$

Use initial conditions to find C and conclude that, whenever α has support in $[0, \tau_p]$, the slice selection profile $m_{xy}(\omega, \tau_p)$ will be equal to the product of $2\pi i \gamma M_z^0 e^{-i\omega\,\tau_p}$ and $\mathcal{F}_{ang}^{-1}\{\alpha(\omega)\}$, where $\mathcal{F}_{ang}$ indicates the angular Fourier transform.

Pulse Sequence

Problem 3. Draw a pulse sequence for a given k-space trajectory in a fixed slice. Here, we will assume that the duration of the entire trajectory is still less than $T1$ or $T2$ times. Make sure you mark clearly all points from 0 to 8 on your pulse sequence.

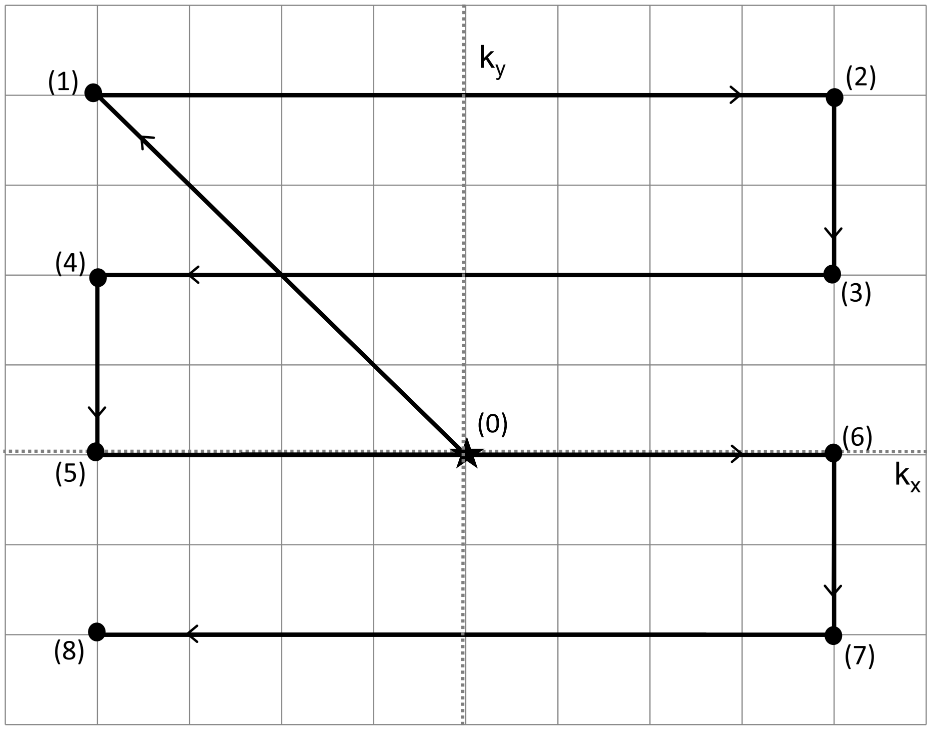

FIGURE 1. k-space trajectory.