ASSIGNMENT 8: SAMPLING AND THE ROTATING FRAME

Shannon Sampling Theorem

This week in class we discussed **Shannon-Kotel'nikov-Whittaker Sampling Theorem**, which states that a square-summable, $L-\eta$ bandlimited function f can be reconstructed from its samples with spacing $\frac{1}{2L}$ by the use of an (L,η)-window function φ as follows

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2L}\right) \varphi\left(x - \frac{n}{2L}\right).$$

We will look at the particular case when $\varphi(x) = \operatorname{sinc}(x)$.

1. Suppose $h \in L^2(\mathbb{R})$ and $\operatorname{supp} h \subseteq [-L, L]$. (So h is L-bandlimited, such a function is also called of exponential type.) Suppose also that Fourier inversion holds at every point for h, that is, $(\hat{h})^\vee(x) = h(x)$ for all $x \in \mathbb{R}$.

a. Prove that for $|\xi| \le L$, we have

$$\hat{h}(\xi) = \sum_{n\in\mathbb{Z}} \frac{\pi}{L}\, h\left(\frac{n\pi}{L}\right) e^{-in\pi\xi/L}. \tag{1}$$

Hint: recall from Assignment 5, exercise 6 that $\{e^{-2\pi i n y}\}_{n\in\mathbb{Z}}$ is a complete orthonormal set in $L^2([0,1])$. Renormalize this set so it would form an o.n. basis for $L^2([-L,L])$. Now expand $\hat{h}$ in terms of this new set. Note that in the equation

$$\left\langle \hat{h}, e^{-in\pi t/L} \right\rangle = \frac{1}{2L} \int_{-L}^{L} \hat{h}(y)\, e^{in\pi y/L}\, dy,$$

the interval of integration can be replaced by $\mathbb{R}$ because of the exponential type/bandlimiting assumption. Next recall the Triple Smiley Theorem from Week 3 Prop.1.10, part (a) - Decomposition, and apply that to $\hat{h}(\xi)$ to obtain (1). One might want to recall the 'angular' Fourier Transform and inversion, i.e., no 2π in the exponent. Then, for example, the 'angular' Fourier Inversion is

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\xi)\, e^{ix\xi}\, d\xi. \tag{2}$$

b. Prove that

$$h(x) = \sum_{n\in L\mathbb{Z}} h\left(\frac{n\pi}{L}\right) \frac{\sin(Lx - n\pi)}{Lx - n\pi}.$$

Hint: Recall the 'angular' Fourier inversion (2). By the exponential type or bandlimiting assumption, the integral on $\mathbb{R}$ can be replaced by $\int_{-L}^{L}$. Substitute equation (1) into this expression.

Rotation of cross-product

2. Prove that if $\boldsymbol{W}$ is a proper rotation (orthogonal change-of-variables matrix), that is, a matrix that preserves inner-products and has determinant 1, then for any $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$, we have

$$\boldsymbol{W}[\boldsymbol{a} \times \boldsymbol{b}] = [\boldsymbol{W}\boldsymbol{a}] \times [\boldsymbol{W}\boldsymbol{b}].$$

Remark. Intuitively, the statement must be true, since the cross-product has a geometric description which is invariant to proper rotations of the axes, but to directly prove this result using algebra would be difficult and quite involved. In fact, the reason we need the result in the first place is that the algebra involved in showing that this holds even for a particular rotation, like the rotating frame change-of-axes matrix $\boldsymbol{W}(t)$, quickly becomes unmanageable.

Sketch of the solution. To prove the claim, we will use the fact that a vector in $\mathbb{R}^3$ is fully determined by its inner product with the other vectors in $\mathbb{R}^3$. So, our goal will be to show that for any $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c} \in \mathbb{R}^3$, the following equality holds:

$$\boldsymbol{c} \cdot (\boldsymbol{W}[\boldsymbol{a} \times \boldsymbol{b}]) = \boldsymbol{c} \cdot ([\boldsymbol{W}\boldsymbol{a}] \times [\boldsymbol{W}\boldsymbol{b}]).$$

Products like those above are called 'triple products'.

Do this in two steps. **First**, show that a triple product $\boldsymbol{c} \cdot \boldsymbol{a} \times \boldsymbol{b}$ is equal to the determinant $|\boldsymbol{cab}|$, that is, the determinant of the matrix whose columns are $\boldsymbol{c}$, $\boldsymbol{a}$, and $\boldsymbol{b}$.

Secondly, use the fact that if $\boldsymbol{W}$ is a rotation, $|\boldsymbol{W}^{-1}| = 1$ and the fact that the determinant of a product is the product of the determinants to take $\boldsymbol{W}^{-1}$ inside the determinant. Finish the argument using the fact that $\boldsymbol{W}$ and $\boldsymbol{W}^{-1}$ preserve inner products.

The Bloch Equation in the Rotating Reference Frame

The $\boldsymbol{B}_0$ field is by far the strongest part of the magnetic field in MRI, making the behavior of the bulk magnetization $\boldsymbol{M}(r,t)$ under this field alone a rough short-term approximation to their overall behavior. In the absence of relaxation terms, the solution to the Bloch equation without relaxation terms (for example, the equation (2) in Assignment 7 with $\boldsymbol{B} = \boldsymbol{B}_0$) is

$$\boldsymbol{M}(r,t) = \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t & 0 \\ -\sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{M}(r,0) \equiv \boldsymbol{U}(t)\boldsymbol{M}(r,0). \tag{3}$$

We will soon want to find the effects of the weaker RF field, which rotates at this same frequency. This problem is greatly simplified by introducing a rotating frame of reference that neutralizes the motion due to $\boldsymbol{B}_0$, and also turns the RF field into a stationary, rather than oscillatory, field. We do this by setting

$$\boldsymbol{W}(t) = [\boldsymbol{U}(t)]^{-1} = \begin{pmatrix} \cos\omega_0 t & -\sin\omega_0 t & 0 \\ \sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

and defining rotating-frame coordinates

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \boldsymbol{W}(t) \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

In the following exercise we obtain the differential equation that describes the behavior of the magnetization in the rotating frame, which we denote by $\boldsymbol{m}(r,t$, and thus, $\boldsymbol{m}(r,t) = W(t)\,\boldsymbol{M}(r,t)$. The behavior of $\boldsymbol{m}$ is given by

$$\boldsymbol{m}'(r,t) = \boldsymbol{m}(r,t) \times \gamma \boldsymbol{B}_{\text{eff}}(r,t) - \frac{\boldsymbol{m}^{\perp}(r,t)}{T_2} + \frac{\boldsymbol{M}^0(r) - \boldsymbol{m}^{\|}(r,t)}{T_1}, \tag{5}$$

where $\boldsymbol{B}_{\text{eff}}(r,t) = \boldsymbol{W}(t)\,\boldsymbol{B}(r,t) - (0,0,\frac{\omega_0}{\gamma})$ and $\boldsymbol{M}^0(r) = (0,0,M^0)$. We will show this as follows:

3. Use the product rule from the previous exercise to expand the derivative of $\boldsymbol{m}(t)$ in terms of $\boldsymbol{W}(t)$, $\boldsymbol{M}(t)$, and their derivatives. Then show that the result is the same as the equation (5) as follows:

a. Show that the terms not involving the external field $\boldsymbol{B}$ in the original Bloch Equation become the terms not involving $\boldsymbol{B}_{\text{eff}}$ in the equation (5).

b. Use Exercise **2.** to show that the term involving $\boldsymbol{B}$ in the original Bloch Equation becomes the term involving $\boldsymbol{B}_{\text{eff}}$ in the equation (5).