ASSIGNMENT 7: KEY

Recall the Bloch equation

$$\frac{d\boldsymbol{M}(r,t)}{dt} = \boldsymbol{M}(r,t) \times \gamma \boldsymbol{B}(r,t) - \frac{\boldsymbol{M}^{\perp}}{T_2} - \frac{\boldsymbol{M}^{\|}(r,t) - \boldsymbol{M}^0(r,t)}{T_1}, \tag{1}$$

Here, $\boldsymbol{B}(r,t)$ is the external magnetic field, the second and the third terms on the right-hand side correspond to the T1 and T2 relaxation processes of bulk magnetization $\boldsymbol{M}(r,t)$. To shorten the notation we will write $\boldsymbol{M}'(r,t)$ instead of $\dfrac{d\boldsymbol{M}(r,t)}{dt}$. First we consider the Bloch equation without the relaxation terms:

$$\boldsymbol{M}'(r,t) = \boldsymbol{M}(r,t) \times \gamma \boldsymbol{B}(r). \tag{2}$$

We would like to show that the solution to this differential equation is

$$\boldsymbol{M}(r,t) = \begin{pmatrix} \cos(\omega(r)t) & \sin(\omega(r)t) & 0 \\ -\sin(\omega(r)t) & \cos(\omega(r)t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{M}(r,0). \tag{3}$$

We do it in three different ways:

 # 1. *Rewrite the equation* (2) *as a system of first order linear differential equations. Then, to get simple one-dimensional derivatives, take the derivative of both sides of each equation, and substitute from the original system to eliminate the first derivatives. At this point, we have two separate second-order linear homogeneous equations. Denote* $\omega_0 = \gamma b_0$. *Recall from elementary differential equations that we can solve such equations by assuming that fundamental solutions are of the form* $\exp(rt)$. *Make this assumption about* $M_1(t)$ *and* $M_2(t)$, *and solve for* r *(this should amount to solving a quadratic equation). Since the equations are second order, we know that there should be exactly two such linearly independent, elementary solutions. As long as the two values of* r *which you got for each of* M_1 *and* M_2 *above were distinct, you have both elementary solutions. If they were not distinct, in general you would use reduction of order, but in this instance the values are distinct (if yours are not, check your work).*

Next, assume that the actual functions M_1 *and* M_2 *are linear combinations of the two fundamental solutions you found for each. The final step is to find the four coefficients (you can make your work slightly easier here by combining terms and using coefficients on sine and cosine rather than on the original exponential functions). Do this using a combination of the original system of first-order linear equations and initial conditions. Your final answer, which should have the coefficients in terms of* $M_1(0)$ *and* $M_2(0)$, *should be equivalent to (3) with* $w(r) = w_0$.

Solution: We can write the equation with the cross product as a system of linear, first-order equations:

$$\begin{aligned} M_1'(t) &= \gamma b_0 M_2(t), \\ M_2'(t) &= -\gamma b_0 M_1(t). \end{aligned}$$

Taking the derivative of both sides each equation above and using that $\omega_0 := \gamma b_0$, we get

$$\begin{aligned} M_1''(t) &= \omega_0 M_2'(t), \\ M_2''(t) &= -\omega_0 M_1'(t). \end{aligned}$$

Then, substituting from the original equations to get rid of the first derivatives, we find that

$$\begin{aligned} M_1''(t) &= -\omega_0^2 M_1(t), \\ M_2''(t) &= -\omega_0^2 M_2(t). \end{aligned}$$

We now proceed as in an elementary differential equations course and assume that the fundamental solutions (before taking linear combinations, as we always can with linear differential equations) are of the form $M_1(t) = \exp(r_1 t)$ and $M_2(t) = \exp(r_2 t)$. (We emphasize again that r_1 and r_2 are constants here.) Making this assumption and substituting into the second-order differential equations above, we get

$$\begin{aligned} r_1^2 \exp(r_1 t) &= -\omega_0^2 \exp(r_1 t), \\ r_2^2 \exp(r_2 t) &= -\omega_0^2 \exp(r_2 t), \end{aligned}$$

so that $r_1,\ r_2 = \pm i\omega_0$. Since this gives two fundamental solutions to each equation, we know that we have found all of them, so we now allow M_1 and M_2 to be linear combinations of these solutions, i.e.,

$$\begin{aligned} M_1(t) &= c_1 \exp(i\omega_0 t) + c_2 \exp(-i\omega_0) \\ &= (c_1 + c_2)\cos\omega_0 t + i(c_1 - c_2)\sin\omega_0 t \\ &= a_1 \cos\omega_0 t + a_2 \sin(\omega_0 t), \end{aligned}$$

and, similarly,

$$M_2(t) = b_1 \cos\omega_0 t + b_2 \sin\omega_0 t.$$

Finally, we need to solve for the constants, using the first-order equations and initial conditions. Substituting the above equations into our original system of first-order differential equations gives us that $a_1 = -b_2$ and $a_2 = b_1$. Finally, since $\cos 0 = 1$ and $\sin 0 = 0$, we get $a_1 = M_1(0)$ and $b_1 = M_2(0)$, so that

$$\begin{aligned} M_1(t) &= M_1(0)\cos\omega_0 t + M_2(0)\sin\omega_0 t, \\ M_2(t) &= M_2(0)\cos\omega_0 t - M_1(0)\sin\omega_0 t \end{aligned}$$

Since $M_3'(t) = 0$ for all t, $M_3(t) = M_3(0) = \textit{const}$ for all t.

 # 2. *Note that the method used above only worked because we were able to take second derivatives and get differential equations that involved M_1 and M_2 in isolation. When we try to solve the general Bloch equation for a constant $\boldsymbol{B}_0$ field, this trick will not work. Nor, for that matter, does it work if we allow $\boldsymbol{B}$ to vary with time, because when we do so, the product rule will introduce extra terms when we try to take the derivative as in Exercise # 1. In this exercise we will use another method that solves systems of linear differential equations.*

Consider a differential equation of the form $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$, where $\boldsymbol{A}$ is a matrix. Assume that fundamental solutions are of the form $\mathbf{x}(t) = \mathbf{x}_0 \exp(rt)$. If so, then by plugging this specific solution into the original equation, we see that

$$r\boldsymbol{x}_0 \exp(rt) = \boldsymbol{A}\boldsymbol{x}_0 \exp(rt),$$

which shows that $\boldsymbol{x}_0$ must be an eigenvector of $\boldsymbol{A}$ with corresponding eigenvalue r. In fact as long as all the eigenvalues of $\boldsymbol{A}$ are distinct, a full set of fundamental solutions can be expressed in this form, so we can solve such a system by finding all the eigenvectors and eigenvalues of A, and then taking an appropriate linear combination to satisfy the initial conditions.

The Bloch equation without relaxation terms (2) is in such a form.Use the method described above to derive the solution operator; that is, find the 3 eigenvalues, which will all be distinct in this case, and the corresponding eigenvectors. Then use the fact that the general solution is a linear combination of the three fundamental solutions given by these eigenvalue/vector pairs, and the initial conditions to solve the system.

Solution: We begin by finding the eigenvalues of the matrix in question:

$$\mathbf{A} := \gamma \begin{pmatrix} 0 & b_0 & 0 \\ -b_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_0 & 0 \\ -\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking the determinant of $\boldsymbol{A} - r\boldsymbol{I}$, we get the characteristic equation,

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = -r^3 - \omega_0^2 r = -r(r^2 - \omega_0^2),$$

which has roots $r = 0$ and $r = \pm i\omega_0$, all of multiplicity one (which multiplicity is this geometric or algebraic?). Next we find the corresponding eigenvectors by looking at the constraints that the equation $(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$ places on x_1, x_2, and x_3. By writing out each component of the product and setting it equal to zero, we get the following (note that $\boldsymbol{x}$ is determined only up to a constant multiple – this is always the case with eigenvectors):

- when $r = 0$, $x_1 = x_2 = 0$ and x_3 is arbitrary, so $\boldsymbol{x} = a(0, 0, 1)$ for any $a \in \mathbb{R}$,
- when $r = i\omega_0$, $x_2 = ix_1 = 0$ and $x_3 = 0$, so $\boldsymbol{x} = b(1, i, 0)$ for any $b \in \mathbb{R}$,
- when $r = -i\omega_0$, $x_2 = -ix_1 = 0$ and $x_3 = 0$, so $\boldsymbol{x} = c(1, -i, 0)$ for any $c \in \mathbb{R}$.

Thus, our general solution is

$$\begin{aligned} \boldsymbol{M}(t) &= a(0, 0, 1) + be^{i\omega_0 t}(1, i, 0) + ce^{-i\omega_0 t}(1, -i, 0) \\ &= (be^{i\omega_0 t} + ce^{-i\omega_0 t}, i(be^{i\omega_0 t} - ce^{-i\omega_0 t}), a). \end{aligned}$$

From our initial conditions we find that $b + c = M_1(0)$ and that $i(b - c) = M_2(0)$, so $b = \frac{1}{2}(M_1(0) - iM_2(0))$, $c = \frac{1}{2}(M_1(0) + iM_2(0))$, and $a = M_3(0)$. Converting the complex exponentials to trigonometric functions using Euler's formula and canceling terms, we find that

$$\mathbf{M}(t) = \begin{pmatrix} M_1(0)\cos\omega_0 t + M_2(0)\sin\omega_0 t \\ M_2(0)\cos\omega_0 t - M_1(0)\sin\omega_0 t \\ M_3(0) \end{pmatrix}$$

 # 3. *This final method yields the same solution but in a different form, the one we will use later in order to describe the actual measurements being taken in MRI and the process used to reconstruct the image from those measurements. In MRI, we often treat the transverse part of $\mathbf{M}(t)$, the x and y components, as a single complex number rather than a vector in $\mathbb{R}^2$. To this end, define the transverse component in complex form,*

$$M_{xy}(t) := M_1(t) + iM_2(t). \tag{4}$$

Now, note that in the Bloch phenomenological equation the longitudinal and transverse components of the bulk magnetization do not interact, we can analyze M_3 and the complex *transverse component $M_{xy}(t)$ in separate differential equations. We will take for granted from Exercises # 1 and # 2 above that $M_3(t) = M_3(0)$ is the third component solution, and solve for M_{xy} in the complex form (4).*

Write out the two equations that M_1 and M_2 satisfy. By linearity of the derivative, we have

$$M'_{xy}(t) = M'_1(t) + iM'_2(t).$$

Combine this with the differential equations, and show that the right side of the above equation can be rewritten as the product of a complex scalar and $M(t)$.

As usual for linear differential equations, solve it either by assuming the ansatz

$$M_{xy}(t) = a\, e^{rt},$$

and then deriving r based on the equation, or by using separation of variables. Then, use the initial conditions to show that if we define

$$\varphi = \arctan\left(\frac{M_2(0)}{M_1(0)}\right), \quad i.e., \quad \frac{M_2(0)}{M_1(0)} = \tan\varphi,$$

a solution for the transverse components in complex form is

$$M_{xy}(t) = |M_{xy}(0)|\; e^{i(-\omega_0 t+\varphi)}.$$

Solution: In order to analyze the transverse component as a complex function, define the complex version of the first two components of $\mathbf{M}(t)$,

$$M_{xy}(t) := M_1(t) + iM_2(t).$$

Using the Bloch equation and linearity of the derivative, we see that

$$\begin{aligned} M'_{xy}(t) &= M'_1(t) + iM'_2(t) \\ &= \omega_0 M_2(t) - i\omega_0 M_1(t) \\ &= -i\omega_0(M_1(t) + iM_2(t)) \\ &= -i\omega_0 M(t). \end{aligned}$$

Making the usual assumption for linear differential equations that $M_{xy}(t) = a\, e^{rt}$ — where, in this context, we expect M_{xy} to be complex—we find that $M'_{xy}(t) = rM_{xy}(t)$ and so $r = -i\omega_0$. Then, based on the initial condition that $M_{xy}(0) = M_1(0) + iM_2(0)$, we see that $a = M_1(0) + iM_2(0) = |M_{xy}|\,(t)e^{i\varphi}$, where $\varphi = \arctan\dfrac{M_2(0)}{M_1(0)}$. Thus, writing the solution as $\mathbf{M}(t) = (M_1(t) + iM_2(t), M_3(t))$, the solution to the Bloch equation without relaxation terms is

$$\mathbf{M}(t) = \left(|\mathbf{M}_{xy}(0)|\; e^{i(-w_0 t+\varphi)}, M_3(0)\right)^t.$$

 # 4. Recall the Bloch equation with relaxation terms:

$$\boldsymbol{M}'(r,t) = \boldsymbol{M}(r,t) \times \gamma\boldsymbol{B}(r,t) - \frac{\boldsymbol{M}^{\perp}(r,t)}{T_2(r)} - \frac{\boldsymbol{M}^{\parallel}(r,t) - \boldsymbol{M}^0(r,t)}{T_1(r)}. \tag{5}$$

The solution to the free precession problem in general is

$$\begin{cases} \boldsymbol{M}_{xy}(r,t) &= e^{-t/T_2(r)} \begin{pmatrix} \cos(\omega(r)t) & \sin(\omega(r)t) \\ -\sin(\omega(r)t) & \cos(\omega(r)t) \end{pmatrix} \boldsymbol{M}_{xy}(r,0), \\ M_z(r,t) &= (1 - e^{-t/T_1(r)})M_z^0(r) - e^{-t/T_1(r)} M_z(r,0) \end{cases} \tag{6}$$

where $\boldsymbol{M}_{xy}(r,t) = (M_x(r,t), M_y(r,t))$.

Either of the methods used in exercises # 2 or # 3 can be used to solve this problem; we will use the former in this case, and the latter in the assignment on excitation (it is important to have a complex-valued solution in order to design selective excitation pulses).

Note that the Bloch equation with relaxation terms, although affine, is not quite linear (meaning that it is not homogeneous in M_3, there is a constant term present) since the derivative of the M_3 is a product of a scalar and $M_3^0 - M_3$, rather than M_3 itself. This prevents us from writing the equation in matrix form. Make a change of variables in the third coordinate to correct this.

Now, solve the differential equation as in exercise # 2, by finding the eigenvalues and eigenvectors of the equation and using initial conditions to determine the appropriate linear combination of terms.

Solution: We begin by finding the eigenvalues of the matrix in question,

$$\mathbf{A} := \begin{pmatrix} -\frac{1}{T_2} & \omega_0 & 0 \\ -\omega_0 & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{pmatrix}.$$

Taking the determinant of $\boldsymbol{A} - r\boldsymbol{I}$ (expanding along the third minor, so that all but one term is zero) we get the characteristic equation,

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = -\left(\frac{1}{T_1} + r\right)\left(\left(\frac{1}{T_2} + r\right)^2 + \omega_0^2\right),$$

which has roots $r = -\frac{1}{T_1}$ and $r = -\frac{1}{T_2} \pm i\omega_0$ all of multiplicity one. Next we find the corresponding eigenvectors by looking at the constraints that the equation $(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$ places on x_1, x_2, and x_3. By finding the kernel of $A - rI$ for each of these r values, we find that:

- when $r = -\frac{1}{T_1}$, $x_1 = x_2 = 0$ and x_3 is arbitrary, so $\boldsymbol{x} = a(0,0,1)$ for any $a \in \mathbb{R}$,
- when $r = -\frac{1}{T_1} + i\omega_0$, $x_2 = ix_1 = 0$ and $x_3 = 0$, so $\boldsymbol{x} = b(1,i,0)$ for any $b \in \mathbb{R}$,
- when $r = -\frac{1}{T_1} - i\omega_0$, $x_2 = -ix_1 = 0$ and $x_3 = 0$, so $\boldsymbol{x} = c(1,-i,0)$ for any $c \in \mathbb{R}$.

Thus, our general solution is

$$\begin{aligned} \boldsymbol{M}(t) &= a\, e^{-1/T_1}(0,0,1) + be^{-1/T_2 + i\omega_0 t}(1,i,0) + ce^{-1/T_2 - i\omega_0 t}(1,-i,0) \\ &= (be^{-1/T_2 + i\omega_0 t} + ce^{-1/T_2 - i\omega_0 t}, i(b\, e^{-1/T_2 + i\omega_0 t} - c\, e^{-1/T_2 - i\omega_0 t}), ae^{-1/T_1}). \end{aligned}$$

From our initial conditions we find that $b + c = M_1(0)$ and that $i(b - c) = M_2(0)$, so $b = \frac{1}{2}(M_1(0) - iM_2(0)$, $c = \frac{1}{2}(M_1(0) + iM_2(0)$, and $a = M_3(0)$. Converting the complex exponentials to trigonometric functions using Euler's formula and canceling terms, the solution is as desired,

$$\begin{cases} \boldsymbol{M}_{xy}(r,t) &= e^{-t/T_2(r)} \begin{pmatrix} \cos(\omega(r)t) & \sin(\omega(r)t) \\ -\sin(\omega(r)t) & \cos(\omega(r)t) \end{pmatrix} \boldsymbol{M}_{xy}(r,0), \\ M_z(r,t) &= (1 - e^{-t/T_1(r)})M_z^0(r) - e^{-t/T_1(r)} M_z(r,0) \end{cases}$$

where $\boldsymbol{M}_{xy}(r,t) = (M_x(r,t), M_y(r,t))$.