ASSIGNMENT 7: THE BLOCH EQUATION

Let $\boldsymbol{M}(r,t)$ denote the bulk magnetization at the location $r = (x,y,z) \in \mathbb{R}^3$ and time $t \in \mathbb{R}$ with components written either as $(M_1(r,t), M_2(r,t), M_3(r,t))^t$ or $(\boldsymbol{M}^{\perp}(r,t), \boldsymbol{M}^{\|}(r,t))^t$. Recall from lectures the Bloch phenomenological equation

$$\frac{d\boldsymbol{M}(r,t)}{dt} = \boldsymbol{M}(r,t) \times \gamma \boldsymbol{B}(r,t) - \frac{\boldsymbol{M}^{\perp}(r,t)}{T_2} - \frac{\boldsymbol{M}^{\|}(r,t) - \boldsymbol{M}^0(r,t)}{T_1}, \tag{1}$$

Here, $\boldsymbol{B}(r,t)$ is the external magnetic field, the second and the third terms on the right-hand side correspond to the T1 and T2 relaxation processes of the bulk magnetization $\boldsymbol{M}(r,t)$. To shorten the notation we will write $\boldsymbol{M}'(r,t)$ instead of $\dfrac{d\boldsymbol{M}(r,t)}{dt}$.

In this assignment we will obtain solutions of this equation. First, we will ignore the relaxation terms and solve the simplified Bloch equation via 3 different approaches. Then we will restore the relaxation terms and with known methods solve the 'full' Bloch equation.

1. Solving the Bloch Equation in the absence of Relaxation Terms

Assume that the background field $\boldsymbol{B}(r) = (0,0,b(r))$ is constant with respect to time. In this case, the bulk magnetization $\boldsymbol{M}(r,t)$ satisfies the relation

$$\boldsymbol{M}'(r,t) = \boldsymbol{M}(r,t) \times \gamma \boldsymbol{B}(r). \tag{2}$$

We would like to show that the solution to this differential equation is

$$\boldsymbol{M}(r,t) = \begin{pmatrix} \cos(\omega(r)t) & \sin(\omega(r)t) & 0 \\ -\sin(\omega(r)t) & \cos(\omega(r)t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{M}(r,0). \tag{3}$$

We will do this using **three** different methods. The **first** and simplest is to solve a pair of second-order differential equations. This method, though fast, is not flexible enough to handle more complicated cases that appear later, first when we account for relaxation terms and then when we allow the strength of the field to vary with time. So we introduce two other methods. The **second** method, a standard way to deal with ordinary differential equations involving vector-valued methods, is to use matrices and eigenvalues, which we apply to the general Bloch equation in the next section. The **third** one is a trick involving writing the transverse magnetization as a complex-valued function, so that we can use one- variable methods to solve. This approach will help us in the next assignment, when we design a selective excitation pulse.

Before we begin, a remark on notation. Under free precession conditions, $\boldsymbol{B}(r)$ is constant at any given location, thus, there is no loss of generality from the point of view of solving the differential equation if we assume that $\boldsymbol{B}(r) = \boldsymbol{B}_0 = (0,0,b_0)$ does not depend on location. This will simplify notation since it frees up the variable r, which is typically used as a constant in the exponential function when solving differential equations.

We first need to write out the differential equation. Recall the convenient memory trick for writing a cross product: for $\mathbb{R}^3$-valued vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ represent the standard basis vectors for $\mathbb{R}^3$. When taking the "determinant", we treat the basis vectors as if they were scalars. It is easy to check using this that the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is equivalent to the matrix product:

$$\begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Using the matrix product representation of the cross product given above, the Block equation without relaxation terms becomes

$$\boldsymbol{M}'(t) = \gamma \boldsymbol{M}(t) \times \boldsymbol{B} = \gamma \begin{pmatrix} 0 & b_0 & 0 \\ -b_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M_1(t) \\ M_2(t) \\ M_3(t) \end{pmatrix}. \tag{4}$$

 # 1. Write out the above matrix product as a system of first order linear differential equations. Then, to get simple one-dimensional derivatives, take the derivative of both sides of each equation, and substitute from the original system to eliminate the first derivatives. At this point, we have two separate second-order linear homogeneous equations. Denote $\omega_0 = \gamma b_0$. Recall from elementary differential equations[1] that we can solve such equations by assuming that fundamental solutions are of the form $\exp(rt)$. Make this assumption about $M_1(t)$ and $M_2(t)$, and solve for r (this should amount to solving a quadratic equation). Since the equations are second order, we know that there should be exactly two such linearly independent, elementary solutions. As long as the two values of r which you got for each of M_1 and M_2 above were distinct, you have both elementary solutions. If they were not distinct, in general you would use reduction of order[2], but in this instance the values are distinct (if yours are not, check your work).

Next, assume that the actual functions M_1 and M_2 are linear combinations of the two fundamental solutions you found for each. The final step is to find the four coefficients (you can make your work slightly easier here by combining terms and using coefficients on sine and cosine rather than on the original exponential functions). Do this using a combination of the original system of first-order linear equations and initial conditions. Your final answer, which should have the coefficients in terms of $M_1(0)$ and $M_2(0)$, should be equivalent to (3) $w(r) = w_0$.

 # 2. Note that the method used above only worked because we were able to take second derivatives and get differential equations that involved M_1 and M_2 in isolation. When we try to solve the general Bloch Equation for a constant $\boldsymbol{B}_0$ field, this trick will not work. Nor, for that matter, does it work if we allow $\boldsymbol{B}$ to vary with time, because when we do so, the product rule will introduce extra terms when we try to take the derivative as in Exercise # 1. In this exercise we will use another method that solves systems of linear differential equations[3].

Consider a differential equation of the form $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$, where $\boldsymbol{A}$ is a matrix. Assume that fundamental solutions are of the form $\mathbf{x}(t) = \mathbf{x}_0 \exp(rt)$. If so, then by plugging this specific solution into the original equation, we see that

$$r\boldsymbol{x}_0 \exp(rt) = \boldsymbol{A}\boldsymbol{x}_0 \exp(rt),$$

which shows that $\boldsymbol{x}_0$ must be an eigenvector of $\boldsymbol{A}$ with corresponding eigenvalue r. In fact as long as all the eigenvalues of $\boldsymbol{A}$ are distinct, a full set of fundamental solutions can be expressed in this form, so we can solve such a system by finding all the eigenvectors and eigenvalues of A, and then taking an appropriate linear combination to satisfy the initial conditions.

The Bloch equation without relaxation terms is in this form, as shown by (4). Use the method described above to derive the solution operator; that is, find the 3 eigenvalues, which will all be distinct in this case, and the corresponding eigenvectors. Then use the fact that the general solution

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[1] If you took an elementary differential equations course at ASU, the textbook could have been Boyce-DiPrima, this is in Chapter 3.

[2] Section 3.5 in Boyce-DiPrima.

[3] This can be found in Chapter 7 in Boyce DiPrima; in particular Section 7.5.

is a linear combination of the three fundamental solutions given by these eigenvalue/vector pairs, and the initial conditions to solve the system.

 # 3. This final method will yields the same solution in a different form, the one we will use later in order to describe the actual measurements being taken in MRI and the process used to reconstruct the image from those measurements. In MRI, we often treat the transverse part of $\mathbf{M}(t)$, the x and y components, as a single complex number rather than a vector in $\mathbb{R}^2$. To this end, define the transverse component in complex form,

$$M_{xy}(t) := M_1(t) + iM_2(t).$$

Now, note that in the Bloch Phenomenological equation the longitudinal and transverse components of the bulk magnetization do not interact, we can analyze M_3 and the complex transverse component $M_{xy}(t)$ in separate differential equations. We will take for granted from Exercises (1) and (2) above that $M_3(t) = M_3(0)$ is the third component solution, and solve for M_{xy} in this complex form.

Write out the two equations that M_1 and M_2 satisfy. By linearity of the derivative, we have

$$M'_{xy}(t) = M'_1(t) + iM'_2(t).$$

Combine this with the differential equations, and show that the right side of the above equation can be rewritten as the product of a complex scalar and $M_{xy}(t)$.

As usual for linear differential equations, solve either by assuming the ansatz

$$M_{xy}(t) = a\, e^{rt},$$

and then deriving r based on the equation, or by using separation of variables. Then, use the initial conditions to show that if we define

$$\varphi = \arctan\left(\frac{M_2(0)}{M_1(0)}\right),$$

a solution for the transverse components in complex form is

$$M_{xy}(t) = |M_{xy}(0)|\; e^{i(-\omega_0 t + \varphi)}.$$

2. SOLVING THE GENERAL FREE PRECESSION EQUATION

We return to the Bloch equation with relaxation terms (here, $r = (x, y, z) \in \mathbb{R}^3$ denotes the location):

$$\boldsymbol{M}'(r,t) = \boldsymbol{M}(r,t) \times \gamma \boldsymbol{B}(r,t) - \frac{\boldsymbol{M}^{\perp}(r,t)}{T_2(r)} - \frac{\boldsymbol{M}^{\|}(r,t) - \boldsymbol{M}^0(r,t)}{T_1(r)}. \tag{5}$$

The solution to the free precession problem in general is

$$\begin{cases} \boldsymbol{M}_{xy}(r,t) &= e^{-t/T_2(r)} \begin{pmatrix} \cos(\omega(r)t) & \sin(\omega(r)t) \\ -\sin(\omega(r)t) & \cos(\omega(r)t) \end{pmatrix} \boldsymbol{M}_{xy}(r,0), \\ M_z(r,t) &= (1 - e^{-t/T_1(r)})M_z^0(r) - e^{-t/T_1(r)} M_z(r,0), \end{cases} \tag{6}$$

where $\boldsymbol{M}_{xy}(r,t) = (M_x(r,t), M_y(r,t))$.

 # 4. Either of the methods used in exercises # 2 or # 3 can be used to solve the above equation (5); we will use the former in this case, and the latter in the assignment on excitation (it is important to have a complex-valued solution in order to design selective excitation pulses).

Note that the Bloch equation with relaxation terms, although affine, is not quite linear (meaning that it is not homogeneous in M_3, there is a constant term present) since the derivative of the M_3 is a product of a scalar and $M_3^0 - M_3$, rather than M_3 itself. This prevents us from writing the equation in matrix form. Make a change of variables in the third coordinate to correct this.

Now, solve the differential equation as in exercise # 2, by finding the eigenvalues and eigenvectors of the equation and using initial conditions to determine the appropriate linear combination of terms.