ASSIGNMENT 6: THE FOURIER TRANSFORM

The first several problems focus on the Gaussian, a real-valued function on $\mathbb{R}^n$, defined by $e^{a|x|^2}$ (not normalized form). This function is important not only in Fourier analysis, but in distribution theory, probability, and several other areas of mathematics, also engineering, physics, economics and finance, and plenty others. So it is worth developing some familiarity with its properties. The exercises on the Gaussian involve finding its Fourier transform, which is used in the notes (Week 6) to prove the Fourier inversion formula. We then derive the Fourier transform of the box function, which is important in sampling theory, and calculate some example (angular) Fourier transforms that appear in the Epstein textbook.

Observe that Gaussian is a Schwartz space function (it is C^∞ and rapidly decaying). The Fourier transform maps Schwartz space into itself, i.e., the Fourier transform of a Schwartz function will also be infinitely smooth and rapidly decaying (as we have seen on the Gaussian). However, the Fourier transform is also well defined for L^1 functions, which also decay at infinity but not as fast as functions from Schwartz class. To which space does the Fourier transform maps L^1 functions? We investigate this question in the next to last problem and conclude with an example how the Fourier transform can be considered for L^2 functions.

1. (The Integral of the Gaussian) Many of the important applications of the Gaussian, both in this course and in probability, involve taking its integral. But this is problematic, because there is no closed form of the Gaussian's antiderivative. We, can, however, compute Gaussian's integral over all of $\mathbb{R}^n$ using a trick involving a change of variables. Following the steps below, show that

$$\int_{\mathbb{R}^n} ce^{-a(x+b)^2}dx = c\left(\sqrt{\frac{\pi}{a}}\right)^n.$$

- **a.** First notice that the integral must be positive, since the integrand always is. Label the integral I. If we find the value of I^2, then this will determine the value of I. Write I^2 as the product of integrals in two different variables, x and y.
- **b.** Combine the product of two integrals, making it a double integral (this is valid because the variables are independent).
- **c.** Look carefully at the resulting double integral. Find a change of variables that will allow you to compute it directly. Take the square root of the result to find the integral of e^{-x^2}.
- **d.** Use the results above to argue that

$$\int_{-\infty}^{\infty} c\, e^{-a(x+b)^2}\, dx = c\sqrt{\frac{\pi}{a}}.$$

- **e.** Finally, conclude that

$$\int_{\mathbb{R}^n} ce^{-a(x+b)^2}dx = c\left(\sqrt{\frac{\pi}{a}}\right)^n.$$

2. (Dilations of Gaussian) We showed in class that $\hat{G} = G$ for the Gaussian defined by $G(x) = e^{-\pi x^2}$, $x \in \mathbb{R}$ (recall the ODE approach). Show that $\widehat{G_\delta}(\xi) = e^{-\pi\delta\xi^2}$ for the dilated Gaussian defined as $G_\delta(x) = \frac{1}{\sqrt{\delta}}\, e^{-\pi x^2/\delta}$.

Hint: We could use the proof from the first exercise to find the Fourier transform of this one-dimensional Gaussian, but instead use change of variables in $\hat{G} = G$.

3. Define

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

Show that $\widehat{\chi_{[-L,L]}}(\xi) = 2L\,\mathrm{sinc}(2L\xi)$.

4. Although its theoretical properties are not as nice because the scaling is off, the angular Fourier transform, defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\cdot\xi}dx,$$

has an advantage in that it is slightly less tedious to compute by hand. Calculate the following one-dimensional angular Fourier transforms (the graphs of these are pictured in Epstein's introductory section on Fourier transform, p.99):

a. Find $\widehat{\chi_{[a,b]}}$. (The book has a typo for this answer.)

b. Show that the Fourier Transform of the tent function f given by

$$f(x) = \begin{cases} 1+x, & x \in [-1,0) \\ 1-x, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

is

$$\hat{f}(\xi) = \frac{2}{\xi^2}(1-\cos\xi).$$

c. Show that the Fourier Transform of the half tent f given by

$$f(x) = \begin{cases} 1-x, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

is

$$\hat{f}(\xi) = \frac{1}{\xi^2}(1-\cos\xi) + \frac{i}{\xi^2}(\sin\xi - \xi).$$

d. Calculate the Fourier Transform of the following function: $f(x) = \chi_{[-2,-1]}(x) - \chi_{[1,2]}(x)$.

5. (Symmetry of the Fourier Transform) We have not discussed the symmetry properties of the Fourier transform in the notes, so we will here. Note that the tent is a real-valued even function, the half-tent is neither even nor odd, and the very last function was odd and real-valued. The Fourier Transform of the tent was entirely real, the transform of the half-tent was complex, and the transform of the last function was entirely imaginary. If you look at the real-valued functions on page 99 of the textbook, you will find that this pattern holds in general; the transforms of even real functions are real, and the transforms of odd real functions are imaginary. Look at the even/odd properties of the kernel of the Fourier Transform, the complex exponential, and explain informally but clearly why this makes sense.

6. (The Fourier Transform on L^1.) Recall that if $f \in L^1(\mathbb{R})$, then it does NOT guarantee that $\hat{f} \in L^1(\mathbb{R})$, one can only show that $\hat{f} \in L^\infty(\mathbb{R})$ (how?).

Do there exist functions such that both f and $\hat{f} \in L^1(\mathbb{R})$ at the same time?

Hint: Try to produce a class of such functions:

- take f which is twice continuously differentiable and has compact support – the notation for this class is $C_c^2(\mathbb{R})$. Show that $|\hat{f}(\xi)| \leq \frac{c}{|\xi|^2}$ (use the integration by parts twice).

- conclude that $\hat{f} \in L^1(\mathbb{R})$ (why was it important to have twice differentiable function ?).

- can you give a specific example of such a function?

7. (The Fourier Transform on $L^1 \cap L^2$.) Prove that if $f, \hat{f} \in L^1(R)$, then $f, \hat{f} \in L^2$.

Hint: Write L^2-norm of $\hat{f}$ and use Hölder's inequality with $(p,p') = (1,\infty)$ pair.