494/598 Fall 2008

ASSIGNMENT 5

Operations with distributions.

1. In class we will discuss what weak (or distributional) derivatives are and that will tell us how to differentiate a distribution.
   (a) Find a derivative of the Dirac delta function, i.e., what is δ'?
   (b) What is δ'' ?

2. Now we will learn how to multiply distributions by a C^∞ function. We say that a function f is a *slowly increasing* C^∞ function, and write $f \in C^\infty_{slow}$, if the function itself and all its derivatives are at most of polynomial growth.
   Let $g \in C^\infty_{slow}$. Find $g \cdot \delta$.

3. Recall that a translation operator is $R_k f(x) = f(x-k)$. What is $R_k\delta$?

Types of convergence.

4. Consider $f_n(x) = \frac{1}{n}\chi_{[0,n]}(x)$. Check if this sequence of functions converges and if yes, find a proper limit:
   (a) pointwise
   (b) uniformly
   (c) in $L^1(\mathbb{R})$ norm.

L^p spaces.

5. Recall that if a domain D is finite, then L^p spaces can be embedded as follows:

$$L^1(D) \subset L^2(D) \subset ...L^p(D)...L^\infty(D).$$

This is not the case when D is infinite. Find a function which is
   (a) in $L^1(R)$ but not in $L^2(R)$,
   (b) in $L^2(R)$ but not in $L^1(R)$.

Fourier orthonormal system.

6. Define the *Fourier system* to be the set of functions $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ on $L^2([0,1])$. Show that the Fourier system is an orthonormal set in $L^2([0,1])$.

   Hint: Consider the inner products $< e^{2\pi inx}, e^{2\pi imx} >$.

BONUS: can you find a mistake?

7. Find the logical error in the following argument:

> Let S be any union of open intervals containing $\mathbb{Q}$. Let x be any element of $\mathbb{R}$. If $x \in \mathbb{Q}$, then obviously $x \in S$, so assume $x \notin \mathbb{Q}$. Pick any positive number ν, no matter how small. Then there is a rational (in fact, there are infinitely many rationals) q such that $x \in B_\nu(q)$. The set S includes all the rationals, so that $x \in S$. Since both S and x were arbitrary, then any union of open intervals which contains all of the rational numbers must also contain all of the real numbers, and thus the rational numbers cannot possibly have measure zero.