

ASSIGNMENT 4: KEY

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are said to be *equivalent* if there are positive constants m and M such that

$$m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1$$

for all $x \in X$.

In class we showed that the relation \sim defined by $\|\cdot\|_1 \sim \|\cdot\|_2$ is an equivalence relation on the set of norms on X , i.e., it has the following properties: reflexivity, symmetry, and transitivity.

I will write this out again for the sake of completeness of this assignment.

Recall that if $a \sim b$, then this equivalence relation is well-defined if it is reflexive ($a \sim a$), symmetrical (if $a \sim b$, then $b \sim a$), and transitive (if $a \sim b$ and $b \sim c$, then $a \sim c$).

- (Reflexivity) Obviously $\|x\|_1 \sim \|x\|_1$ with $m = M = 1$.
- (Symmetry) Suppose $m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1$. Since $m, M > 0$, it follows that $\frac{1}{m}, \frac{1}{M} > 0$, and that

$$\frac{1}{M} \|x\|_1 \leq \|x\|_2 \leq \frac{1}{m} \|x\|_1.$$

- (Transitivity) Let $M_1, M_2, m_1, m_2 \in \mathbb{R}$ satisfy

$$m_1 \|x\|_1 \leq \|x\|_2 \leq M_1 \|x\|_1$$

and let

$$m_2 \|x\|_2 \leq \|x\|_3 \leq M_2 \|x\|_2.$$

Then, putting the inequalities together, we have

$$m_2 m_1 \|x\|_1 \leq m_2 \|x\|_2 \leq \|x\|_3$$

and

$$\|x\|_3 \leq M_2 \|x\|_2 \leq M_2 M_1 \|x\|_1.$$

Since $m_2 m_1$ and $M_2 M_1$ are positive, this shows transitivity.

Now we proceed to the exercises.

1. Show that equivalent norms on X define the same ‘topology’. In other words, prove that a set which is open under some norm is open under any equivalent norm.

SOLUTION: Let $\|x\|_1 \leq M \|x\|_2$ for all $x \in X$. Let G be open in $\|\cdot\|_1$, and let $x \in G$. Then there is an $\epsilon > 0$ such that $\{y : \|y - x\|_1 < \epsilon\} \subset G$. Then for any $y \in X$ such that $\|x - y\|_2 < \epsilon/M$, we have

$$\|x - y\|_1 \leq M \|x - y\|_2 < \epsilon,$$

so that $y \in G$. Thus, G is by definition open under $\|\cdot\|_2$.

Next, we review the Fatou’s Lemma:

Theorem 1. Let $\{f_n\}$ be a sequence of measurable functions with $f_n \geq 0$. Then

$$\liminf_{n \rightarrow \infty} \int f_n \geq \int \liminf_{n \rightarrow \infty} f_n.$$

2. Construct a counter example that if *liminf* is replaced with *limsup*, then the resulting statement of Fatou’s Lemma does not hold.

SOLUTION: Consider the following sequence of functions (all defined on the interval $[0,1]$):

$$\begin{aligned} f_1 &= \chi_{[0,1/2]}, f_2 = \chi_{[1/2,1]}, \\ f_3 &= \chi_{[0,1/4]}, f_4 = \chi_{[1/4,1/2]}, f_5 = \chi_{[1/2,3/4]}, f_6 = \chi_{[3/4,1]}, \\ f_7 &= \chi_{[0,1/8]}, \dots \end{aligned}$$

Then $\limsup_{n \rightarrow \infty} f_n(x) = 1$ for all $x \in [0,1]$ (there will always be a subsequence of functions f_{n_k} such that for a given x the functions $f_{n_k}(x) = 1$). Hence,

$$\int_0^1 \limsup_{n \rightarrow \infty} f_n(x) dx = 1.$$

On the other hand,

$$\begin{aligned} \int_0^1 f_1(x) dx &= 1/2, \int_0^1 f_2(x) dx = 1/2, \\ \int_0^1 f_3(x) dx &= 1/4, \int_0^1 f_4(x) dx = 1/4, \int_0^1 f_5(x) dx = 1/4, \int_0^1 f_6(x) dx = 1/4, \\ \int_0^1 f_7(x) dx &= 1/8, \dots \end{aligned}$$

and thus,

$$\int_0^1 f_n(x) dx \longrightarrow 0.$$

Hence,

$$\limsup_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0.$$

If *liminf* is replaced with *limsup*, then the resulting statement of Fatou's Lemma would state

$$\limsup_{n \rightarrow \infty} \int f_n \geq \int \limsup_{n \rightarrow \infty} f_n$$

which would be false for the given sequence of functions ($0 > 1$).

- 3.** Show that the following ‘‘Reverse Fatou’s Lemma’’ holds: If there exists an integrable function g (this means that $\int |g| < +\infty$) such that $g_n \leq g$ for all n , then

$$\int \limsup_{n \rightarrow \infty} g_n \geq \limsup_{n \rightarrow \infty} \int g_n.$$

SOLUTION: For $n \in \mathbb{N}$ define $f_n = g - g_n$, and note that each $f_n \geq 0$, since $g_n \leq g$. We can apply the Fatou’s Lemma to this new sequence $\{f_n\}$ to obtain

$$\liminf_{n \rightarrow \infty} \int (g - g_n) \geq \int \liminf_{n \rightarrow \infty} (g - g_n),$$

or equivalently,

$$\liminf_{n \rightarrow \infty} \int g + \liminf_{n \rightarrow \infty} \int (-g_n) \geq \int \liminf_{n \rightarrow \infty} g + \int \liminf_{n \rightarrow \infty} (-g_n).$$

Canceling the first integrals on each side and recalling that $\liminf(-f) = -\limsup(f)$, we obtain

$$-\limsup_{n \rightarrow \infty} \int g_n \geq -\int \limsup_{n \rightarrow \infty} g_n,$$

multiplying by -1 both sides, we get

$$\limsup_{n \rightarrow \infty} \int g_n \leq \int \limsup_{n \rightarrow \infty} g_n.$$